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Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function

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Abstract: A systematic approach to the study of semiclassical fluctuations of strings in $AdS_5 \times S^5$ based on the Green-Schwarz formalism is developed. We show that the string partition function is well defined and finite. Issues related to different gauge choices are clarified. We consider explicitly several cases of classical string solutions with the world surface ending on a line, on a circle or on two lines on the boundary of AdS. The first example is a BPS object and the partition function is one. In the third example the determinants we derive should give the first corrections to the Wilson loop expectation value in the strong coupling expansion of the $\mathcal{N} = 4$ SYM theory at large N.

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1. Introduction

The duality between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions is the best studied example of the AdS/CFT correspondence [1, 2, 3]. This duality allows the calculation of gauge theory observables at large N and large 't Hooft coupling from perturbative supergravity or string theory. In particular, Wilson loops are described by classical strings that end at the boundary of AdS [4].

To extend this duality beyond the supergravity limit it is necessary to learn how to handle strings on this space. Because the background includes a Ramond-Ramond 5-form flux, it is difficult to use the RNS formalism to quantize strings in this geometry. Therefore, one is led to use the Green-Schwarz (GS) action. The first step in this direction was the construction of the classical GS action for strings on this background [5].

Even in flat space the GS action is hard to quantize, except in the light cone gauge. However, this action is perfectly applicable to the perturbative analysis of quantum corrections around a non-trivial "long string" classical solution (assuming the classical bosonic background makes the fermionic kinetic term well-defined). This strategy can be applied in either flat or curved space, and, in particular, is well suited for strings in $AdS_5 \times S^5$ where there is a natural static solution [4] to expand about.

The string 2-d loop expansion in $AdS_5 \times S^5$ is an expansion in powers of $\alpha'/R^2 = \lambda^{-1/2}$, where R is the radius parameter of $AdS_5 \times S^5$ and λ is the 't Hooft coupling: the leading term coming from the classical action is proportional to $\sqrt{\lambda}$, the 1-loop correction is just a number, the 2-loop correction will be multiplied by $\alpha'/R^2 = \lambda^{-1/2}$, etc.

The main goal of this paper is to develop technical tools necessary to do calculations of quantum string corrections in $AdS_5 \times S^5$, at least in the one-loop approximation. This is an important step in the extension of the AdS/CFT correspondence beyond the classical level. Our main motivation is to find the quantum string correction to the Wilson loop expectation value, in particular, the first sub-leading (i.e. λ-independent) correction to the quark anti-quark potential.

This problem was first addressed in [6], where the relevant fermionic operator coming from GS action was derived. An important next step was made in [7], where the partition function was expressed in terms of operators defined with respect to the induced 2-d geometry. Refs. [8, 9, 10] also discussed corrections to the quark anti-quark potential in $AdS_5 \times S^5$ and in other related geometries. However, all these previous attempts were incomplete as they encountered problems with divergences, gauge fixing, and other subtleties.

Our aim is to clarify some of these issues and to set up a consistent framework for performing the semiclassical calculations for the GS string in a curved target space. In particular, we shall explain how the divergence proportional to the world sheet curvature $R^{(2)}$ found in [7] is canceled (the cancellation of this divergence in the one-loop approximation in curved target space is essentially the same as in flat space). We will also explain the close relation between the fermionic operators in [6] and in [7] (they correspond to two choices of κ-symmetry gauge).

The paper is organized as follows.

We start with some general comments about the Green-Schwarz action in flat space, and, in particular, how to use it to calculate quantum corrections to a classical solution. This involves gauge fixing, and determining the measure in the path integral. We will find it most reliable to use conformal gauge, where the path integral measure is best understood [11, 12]. Since the theory is critical, the conformal anomalies and, therefore, the 2-d divergences, cancel out. The same mechanism is responsible for the cancellation of leading-order (one-loop) divergences (that are proportional to $R^{(2)}$) in curved space as well.

In section 3 we turn to strings on $AdS_5 \times S^5$. We review the corresponding Brink-DiVecchia-Howe-Polyakov type GS action and explain how to evaluate the quadratic fluctuations around a classical solution. We also comment on the approach based on the Nambu-Goto type action in the static gauge. With careful account of ghosts (and path integral measures) the two approaches should give the same results.

We show that as in [13]–[17] a local Lorentz rotation of GS spinors allows one to systematically transform the quadratic fermionic term in the GS action into the action for a set of 2-d fermions. The problem of computing the partition function is then reduced to the evaluation of determinants of some bosonic and fermionic operators on a 2-d world sheet with an induced metric that is asymptotic to AdS_2.

We study three special examples. In section 4 we consider a string world surface that ends on a single straight line at the boundary of AdS_5. The induced metric on the world surface is that of AdS_2 and the quantum fluctuation fields fit nicely into supersymmetry multiplets on that space. We compute the corresponding vacuum energy and show that it vanishes using a ζ-function regularization. The vacuum energy is related to the partition function by a conformal anomaly. Using that we show that the partition function is equal to one.

Another case which leads again to AdS_2 for the induced 2-d geometry is a circular Wilson loop, which we study in section 5. We comment on the difference between the circular and the straight line cases.

In section 6 we turn to the case of most interest, the surface corresponding to the quark-anti-quark system. Here the induced geometry is more complicated but is still asymptotically AdS_2. We derive the general expression for the partition function (demonstrating in the process the equivalence of the two κ-symmetry gauges $\theta^1 = \theta^2$ and $\theta^1 = i\Gamma_4\theta^2$) and discuss evaluation of the numerical coefficient in the corresponding one-loop correction to the $1/L$ potential using a crude approximation to the geometry.

We summarize our results in section 7.

Some general remarks and explicit calculations are given in appendices. In appendix A we review how a determinant of a Laplace operator changes under rescaling of the measure of the fields. The resulting general relations are useful in computing various contributions to the partition function.

In appendix B we present two different calculations of the partition function in the case of AdS_2 as the induced geometry.

Some comments about the fermionic 2-d determinants related to GS action are given in appendix C.

In appendix D we point out that the expression for the superstring partition function in $AdS_3 \times S^3$ with RR 2-form background is very similar to the one in the $AdS_5 \times S^5$ case.

Below we shall use the following notation: $i, j, \ldots = 0, 1$ and $\alpha, \beta, \ldots = 0, 1$ will denote 2-d world and tangent space indices; $a, b, \ldots = 0, \ldots, 4$ and $p, q, \ldots = 1, \ldots, 5$ will be the tangent space indices of AdS_5 and S^5; $\hat{a} = 0, 1, \ldots, 9$ will be the tangent space indices of the 10-d space-time.

2. Green-Schwarz action in flat space

Before plunging into discussion of strings in curved target space, it is useful to clarify several general points about the GS action. The flat space GS action of type-IIB theory is [18]

$$S_{\text{flat}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \left[-\frac{1}{2}\sqrt{g} g^{ij} \eta_{\hat{a}\hat{b}} \left(\partial_i x^{\hat{a}} - i\bar{\theta}^I \Gamma^{\hat{a}} \partial_i \theta^I\right) \left(\partial_j x^{\hat{b}} - i\bar{\theta}^J \Gamma^{\hat{b}} \partial_j \theta^J\right) - \right.$$
$$\left. - i\epsilon^{ij} s^{IJ} \bar{\theta}^I \Gamma_{\hat{a}} \partial_j \theta^J \left(\partial_i x^{\hat{a}} - \frac{1}{2} i\bar{\theta}^K \Gamma^{\hat{a}} \partial_i \theta^K\right)\right], \qquad (2.1)$$

where $\hat{a} = 0, 1, \ldots, 9$, s^{IJ} is defined by $s^{11} = -s^{22} = 1$, $s^{12} = s^{21} = 0$, g_{ij} $(i, j = 0, 1)$ is a world-sheet metric with signature $(-+)$, $g = -\det g_{ij}$, and θ^I are two left 10-d Majorana Weyl spinors.

This action can be considered in either the Polyakov form, with independent 2-d metric (which can be quantized in the conformal gauge) or in the Nambu form, with the induced metric (which can be quantized in the static gauge). When doing semiclassical expansion near a "long string" configuration it may seem natural to use the Nambu formulation, choosing a static gauge. However, the meaning of conformal invariance conditions and the definition of the path integral measure are clear only in the Polyakov formulation. In that case, the 2-d metric, which at the classical level is proportional to the induced metric, should be treated as independent of the coordinates in checking the conformal invariance constraints on the background target space fields (in particular, in proving that conformal anomalies cancel in flat $D = 10$ space). In the leading 1-loop approximation the Polyakov and Nambu formulations are expected to produce equivalent expressions for the partition function. However, the precise way the divergences cancel may become rather obscure once one sets the metric to be equal to the induced metric, since $\int R^{(2)}$ and $\int \partial x \partial x$ divergences may get mixed up. In particular, the $\int R^{(2)}$ divergences become equivalent to total derivative contributions to x-dependent divergences and reduce to boundary terms (which may eventually cancel against boundary counterterms).

Another important point concerns the distinction between the fermionic kinetic term for GS fermions and for standard 2-d Dirac fermions. As was observed in [15, 17] (see also [19, 13, 16, 14, 20]) in the case of a flat target space, one may perform a local target space rotation that transforms the quadratic GS fermion term into the 2-d fermion kinetic term. The resulting jacobian (see, in particular, [14]) depends on the 2-d metric and its contribution explains why the conformal anomaly of a GS fermion is 4 times bigger than that of a 2-d fermion [21] (which is crucial for understanding how conformal anomalies cancel in $D = 10$ GS string). Similar remarks apply in the case of curved target spaces. As we shall explicitly discuss below, in some simple cases (like the straight string in $AdS_5 \times S^5$) the quadratic part of the GS action has already the 2-d fermion form with respect to the curved geometry of induced metric. In other cases one must perform a rotation to express the action in the 2-d fermion form. In the Polyakov formulation with independent 2-d metric, the jacobian of this must be taken into account for consistent cancellation of conformal anomalies. The contribution of this jacobian may be non-trivial also in the Nambu formulation where it may depend on the x-background.

2.1 Quadratic fluctuations near a classical solution

The Green-Schwarz action (2.1) is not quadratic in fermions, and is difficult to quantize. One standard way to proceed is to choose a light cone gauge. Alternatively, one may resort to a perturbative expansion in powers of α' near a particular classical solution. Since the latter strategy is the only one available in the curved $AdS_5 \times S^5$ case, we shall employ it below. We concentrate on the one-loop approximation, i.e.

on the leading quantum correction to the partition function of the GS string action expanded near a classical solution.

With a suitable choice of coordinates, we can write the "long string" classical solution as

$$x^0 = \sigma^0 \,, \qquad x^1 = \sigma^1 \,. \tag{2.2}$$

The bosonic part of the action (2.1) is simply the Polyakov action, and it can be quantized in the conformal gauge $\sqrt{g}g^{ij} = \delta^{ij}$. This results in 10 massless world-sheet scalars and two ghosts.

Alternatively, one could start with the Nambu form of the action (i.e. first solve for g_{ij} and then quantize the theory). In that case we can again expand near (2.2) and choose the static gauge, i.e. eliminate the fluctuations in $(0,1)$ directions. Then we are left with just eight transverse scalars.

The quadratic term in the fermionic part of the GS action is

$$S_{2\mathrm{F}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \; L_{2\mathrm{F}} = \frac{i}{2\pi\alpha'} \int d^2\sigma \left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}s^{IJ}\right) \bar{\theta}^I \rho_i \partial_j \theta^J \,, \tag{2.3}$$

where g_{ij} can be set equal to η_{ij} in the conformal gauge. ρ_i is the projection of the 10-d Dirac matrices on the world sheet

$$\rho_i \equiv \Gamma_{\hat{a}} \partial_i x^{\hat{a}} = \Gamma_i \,, \tag{2.4}$$

where the last equality is true for the classical solution (2.2). Then

$$L_{2\mathrm{F}} = i\bar{\theta}^1 \Gamma^+ \partial_+ \theta^1 + i\bar{\theta}^2 \Gamma^- \partial_- \theta^2 \,. \tag{2.5}$$

This fermionic action is obviously invariant under $\delta\theta^1 = \Gamma^+\kappa^1$, $\delta\theta^2 = \Gamma^-\kappa^2$ which is just the leading-order term in the κ-symmetry transformation rules

$$\delta_\kappa \theta^I = \rho_i \kappa^{iI} + \cdots \,, \qquad \frac{1}{\sqrt{g}}\epsilon^{ij}\kappa^1_j = -\kappa^{i1} \,, \qquad \frac{1}{\sqrt{g}}\epsilon^{ij}\kappa^2_j = \kappa^{i2} \,. \tag{2.6}$$

Since we can represent the (left subspace, 16-component) 10-d Dirac matrices in the form $\Gamma_i = \tau_i \times I_8$, where τ_i are 2×2 Dirac matrices and I_8 is the 8×8 unit matrix, the meaning of the κ-symmetry transformations in the present case is simply that θ^1 corresponds to eight left 2-d spinors and θ^2 to eight right 2-d spinors.

A natural way to fix κ-symmetry is to set[1]

$$\theta^1 = \theta^2 \equiv \theta \,. \tag{2.7}$$

The remaining degrees of freedom are then 8 real 2-d spinors represented by 16-component left 10-d MW spinor θ.

[1]This gauge (considered also in [22]) is possible only in type-IIB theory where the two spinors have the same chirality. This gauge is also natural in connection with the open string theory — in type-I theory $\theta^1 = \theta^2$ at the boundary of world sheet.

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The global part of the local κ-symmetry transformation of x^μ, that is preserved by the above gauge choice, may then be interpreted as the effective 2-d supersymmetry of the resulting quadratic action, $\delta x^k = \bar\theta^I \Gamma^k \delta_\kappa \theta^I$. This is similar to what happens in the light cone gauge. As we shall see, this simple picture has a direct counterpart in curved case.

2.2 Conformal invariance of GS string in flat space

The proof that the fermionic RNS string is conformally invariant at the quantum level [11] is based on adding together the central charges of all the fields: 1 for each scalar boson, 1/2 for each Majorana 2-d fermion, -26 for the conformal ghosts and 11 for the superconformal ghosts.

Since conformal anomalies are associated with UV divergences, it is not surprising that the same counting is responsible for the cancellation of logarithmic divergences in the properly defined string partition function on a 2-d surface with any number of holes and handles [12]. This is obvious for the scalar and fermion determinants. For the ghosts, the essential extra ingredient is the need to take into account some global factors in the gauge group measure associated with conformal Killing vectors and/or Teichmüller moduli. Then the logarithmic divergences are again proportional to the total central charge times the Euler number of the Riemann surface and cancel out if $D = 10$ (or $D = 26$ in the bosonic string case).

The counting for the GS string is different. We describe here only the one-loop approximation. To discuss the cancellation of conformal anomaly in GS string we need to keep the dependence on a generic fiducial metric g_{ij} in the action (2.3) and in the norms of the fields ($\int d^2\sigma \sqrt{g}\bar\theta\theta$, etc.). Naively, after gauge fixing one gets 10 scalars, 8 Majorana 2-d fermions and the bosonic conformal ghosts. The naive counting would give $10 \times 1 + 8 \times \frac{1}{2} - 26 = -12$. But, in fact, the GS fermionic action depends on the 2-d metric, not as in the case of the standard action for a 2-d spinor (i.e. not through $\sqrt{g}\, e^i_\alpha$ where e^α_i is a zweibein), but rather as a 2-d scalar action (i.e. through $\sqrt{g}\ g^{ij}$). In the conformal gauge $g_{ij} = e^{2\rho}\eta_{ij}$, $e^\alpha_i = e^\rho \delta^\alpha_i$ that effectively results in the replacement of ρ by 2ρ in the conformal anomaly term ($\int \partial\rho\bar\partial\rho$) for a 2-d spinor, giving *four* times bigger a result [21].

Hence the contribution of each of 8 species of GS fermions to the divergence is effectively as of 4 2-d spinors.[2] Then the count of anomalies in the GS string goes

[2] In more detail, the action for a 2-d spinor is $\int d^2\sigma \sqrt{g} g^{ij} \bar\psi e^\alpha_i \tau_\alpha \partial_j \psi$, while for the fermions in the GS action (2.3) the world sheet combination $\tau_i = e^\alpha_i \tau_\alpha$ is replaced by the target space one, $\rho_i = \partial_i x^{\hat a} \Gamma_{\hat a}$. In certain cases the two might be equal, but they do not behave the same way under the conformal transformations of the world-sheet metric. The GS fermions θ are world sheet scalars, so their natural measure is $||\theta||^2 = \int d^2\sigma \sqrt{g}\bar\theta\theta$. In the conformal gauge ($g_{ij} = \sqrt{g}\delta_{ij}$) the GS fermionic action is $\int d^2\sigma \bar\theta \rho_\alpha \partial_\alpha \theta$. Because of the normalization of the θ's, after squaring the fermionic operator we get $\frac{1}{\sqrt{g}}\partial \frac{1}{\sqrt{g}}\bar\partial$. In the case of the 2-d spinors in the conformal gauge the zweibein contributes to the scaling of the action $\int d^2\sigma (\sqrt{g})^{1/2} \bar\psi \tau_\alpha \partial_\alpha \psi$, $||\psi||^2 = \int d^2\sigma \sqrt{g}\bar\psi\psi$. If we rescale ψ to make the

as follows

$$10 - 26 + 8 \times 4 \times \frac{1}{2} = 0\,. \tag{2.8}$$

Essentially equivalent arguments (based on separating the metric dependence in a WZ type jacobian contribution due to a rotation of spinors) which explains why conformal anomaly cancels in $D = 10$ in GS string were given in [15, 17, 14] (see also appendix C).

Since in a covariant regularization the cutoff is coupled to the conformal factor, the cancellation of conformal anomalies should imply also the cancellation of the UV divergences.

2.3 Cancellation of quantum correction to straight string configuration

A natural classical "long string" solution in flat space is (2.2), or restoring the dimensional parameters,

$$x^0 = T\tau\,, \qquad x^1 = L\sigma\,, \qquad \sigma \in \left(-\frac{1}{2}, \frac{1}{2}\right). \tag{2.9}$$

The fluctuations of the $d - 2 = 8$ transverse bosonic coordinates (which are periodic in σ^0 and Dirichlet in σ^1 directions) give for $T \to \infty$ [23]

$$W = -\ln Z = (d-2)W_0\,, \qquad W_0 = \frac{1}{2}[\log\det(-\partial^2)]_{T\to\infty} = -\frac{\pi}{24}\frac{T}{L}\,. \tag{2.10}$$

In the flat-space superstring case this contribution is canceled by the contribution of the fermionic determinant: the total effective number of transverse world-sheet degrees of freedom is equal to zero (as in any flat-space string theory without tachyons [24]) because of the effective 2-d supersymmetry present after choosing $\theta^1 = \theta^2$ and expanding to quadratic order near (2.9). Indeed, the induced metric and zweibein are flat, and thus, apart from the subtlety with cancellation of conformal anomalies discussed above, the GS fermionic determinant is the same as for eight 2-d spinors.

3. Quadratic fluctuations of superstring in $AdS_5 \times S^5$

We now turn to the discussion of the one-loop approximation to the partition function of GS superstring in $AdS_5 \times S^5$. We start with the Polyakov form of GS action

action g-independent as in θ case we get $\int d^2\sigma \bar{\psi}\tau_\alpha\partial_\alpha\psi$, $||\psi||^2 = \int d^2\sigma(\sqrt{g})^{1/2}\bar{\psi}\psi$. The difference compared to θ is now only in the norm. The corresponding 2-nd order operator is $\frac{1}{(\sqrt{g})^{1/2}}\partial\frac{1}{(\sqrt{g})^{1/2}}\bar{\partial}$. It is the difference in the measure factor that now leads to different anomalies: for the operator $f\partial f\bar{\partial}$ the conformal anomaly in the partition function is $\exp(-\frac{1}{48\pi}\int d^2\sigma|\partial\ln f|^2)$, so that the difference between the "scalar" and "2-d spinor" descriptions produces indeed the factor of 4 in the conformal anomaly.

in conformal gauge, expand near a general classical solution, and explicitly check conformal invariance to 1-loop order. We shall also comment on the result obtained by starting with the Nambu-type action and using static gauge. In the following sections we give examples of particular symmetric solutions.

In the context of the AdS/CFT duality, expansion about classical solutions of the string action, namely minimal surfaces, corresponds to computing expectation values of Wilson loop operators in the dual gauge theory [4]. The expectation value of the Wilson loop is given by

$$\langle W \rangle = \int [dx][d\theta][dg]\, e^{-S} \,, \tag{3.1}$$

where S is the string action in $AdS_5 \times S^5$ and the path integral is over all embeddings of the string into $AdS_5 \times S^5$ with proper boundary conditions (the string world surface should end along the loop at the boundary of AdS_5 [4, 25]). Here we assumed the Polyakov form, where, in general, one is to integrate over the moduli of 2-d metrics.

3.1 The action

The bosonic part of the action for a string in $AdS_5 \times S^5$ is[3]

$$S_{\rm B} = \frac{R^2}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ij} G_{\mu\nu}(x) \partial_i x^\mu \partial_j x^\nu \,. \tag{3.2}$$

We have removed the dependence on the AdS scale R ($R^4 = 4\pi\alpha'^2 g_s$) from the space-time metric $G_{\mu\nu}$ ($m = 1, \dots, 4$)

$$ds^2 = \frac{1}{w^2}\left(dw^2 + dx^m dx^m\right) + d\Omega_5^2 \,. \tag{3.3}$$

The leading behavior at large 't Hooft coupling λ is the exponent of the classical action, which is proportional to $R^2/\alpha' = \sqrt{\lambda}$. The string expansion is in inverse powers of $\sqrt{\lambda}$. In most of the paper we set $R = 1$, but it is easy to restore the dependence on R when necessary.

The structure of the full covariant GS string action in $AdS_5 \times S^5$ is rather complicated [5], but the part quadratic in θ^I is simple and is a direct generalization of the quadratic term in the flat-space GS action (2.5)

$$S_{\rm 2F} = \frac{i}{2\pi\alpha'} \int d^2\sigma (\sqrt{g} g^{ij} \delta^{IJ} - \epsilon^{ij} s^{IJ}) \bar\theta^I \rho_i D_j \theta^J \,. \tag{3.4}$$

Here ρ_i are again projections of the 10-d Dirac matrices,

$$\rho_i \equiv \Gamma_{\hat a} E^{\hat a}_\mu \partial_i x^\mu = (\Gamma_a E^a_\mu + \Gamma_p E^p_\mu) \partial_i x^\mu \,, \tag{3.5}$$

[3] For a string representing a Wilson loop of SYM theory and ending at the boundary the classical action is actually a particular Legendre transform of the area [25], but that does not affect the discussion of quantum fluctuations.

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and $E^{\hat{a}}_\mu$ is the vielbein of the 10-d target space metric, $G_{\mu\nu} = E^{\hat{a}}_\mu E^{\hat{b}}_\nu \eta_{\hat{a}\hat{b}}$. The covariant derivative D_i is the projection of the 10-d derivative $D_\mu = \partial_\mu + \frac{1}{4}\Omega^{\hat{a}\hat{b}}_\mu \Gamma_{\hat{a}\hat{b}} - \frac{1}{8\cdot 5!}\Gamma^{\mu_1\ldots\mu_5}\Gamma_\mu \, e^\phi F_{\mu_1\ldots\mu_5}$ ($\Omega^{\hat{a}\hat{b}}_\mu$ is the spin connection and $F_{\mu_1\ldots\mu_5}$ the RR 5-form potential) which appears, e.g., in the Killing spinor equation of type-IIB supergravity. It has the following explicit form [5][4]

$$D_i\theta^I \equiv \left(\delta^{IJ}\mathcal{D}_i - \frac{1}{2}i\epsilon^{IJ}\tilde{\rho}_i\right)\theta^J \,, \qquad \mathcal{D}_i = \partial_i + \frac{1}{4}\partial_i x^\mu \Omega^{\hat{a}\hat{b}}_\mu \Gamma_{\hat{a}\hat{b}} \,, \tag{3.6}$$

where the $\tilde{\rho}_i$ term originates from the coupling to the RR field strength,

$$\tilde{\rho}_i \equiv \left(\Gamma_a E^a_\mu + i\Gamma_p E^p_\mu\right)\partial_i x^\mu \,. \tag{3.7}$$

Note that $\tilde{\rho}_i$ is not identical to ρ_i, unless one is expanding near a classical solution that is constant on S^5.

In general, there is a factor of R^{-1} in front of the 'mass term', so that this term disappears in the flat-space limit.

3.2 Expanding about a classical solution

We first consider the bosonic sector. Expand the Polyakov action (3.2) about a classical solution

$$\begin{aligned} x^\mu \; 6 &\to \bar{x}^\mu + \xi^\mu, \qquad & g_{ij} &\to g_{ij} + \chi_{ij} \,, \\ g_{ij} &= e^{2\lambda}h_{ij} \,, \qquad & h_{ij} &\equiv G_{\mu\nu}(\bar{x})\partial_i\bar{x}^\mu\partial_j\bar{x}^\nu . \end{aligned} \tag{3.8}$$

The classical value of the metric may, in general, differ from the induced metric h_{ij} by an arbitrary conformal factor λ. We fix the 2-d diffeomorphism invariance by imposing the conformal gauge conditions on the fluctuations of the metric

$$\chi_{ij} = \kappa g_{ij} \,, \qquad \text{i.e.} \qquad g_{ij} \to (1+\kappa)g_{ij} \,. \tag{3.9}$$

The remaining conformal degree of freedom of the metric should decouple as in flat 10-d space because of the conformal invariance of type-IIB string theory on $AdS_5 \times S^5$

[4]The 10-d Dirac matrices are split in the '5+5' way, $\Gamma^a = \gamma^a \times I_4 \times \sigma_1$, $\Gamma^p = I_4 \times \gamma^p \times \sigma_2$, where σ_k are Pauli matrices and γ^a, γ^p are 4×4 matrices (corresponding to tangent spaces of AdS_5 and S^5 factors) and I_4 is 4×4 unit matrix (see [5] for details on notation; we use index $p = 1,\ldots,5$ instead of a' in [5]). A $D = 10$ positive chirality 32-component spinor Ψ is decomposed as follows: $\Psi = \psi \times \psi' \times \binom{1}{0}$. In equations written in the 32-component spinor form, θ^I stands for two positive chirality spinors $\binom{\theta^I}{0}$, where θ^I are 16-component spinors used below. The Majorana condition $\bar{\Psi} \equiv \Psi^\dagger\Gamma^0 = \Psi^T\mathcal{C}$, $\mathcal{C} \equiv C \times C' \times i\sigma_2$, then takes the form $\bar{\theta}_{\alpha\alpha' I} \equiv (\theta^{I\beta\beta'})^\dagger(\gamma^0)^\beta_\alpha\delta^{\beta'}_{\alpha'} = \theta^{I\beta\beta'}C_{\beta\alpha}C'_{\beta'\alpha'}$. Here C and C' are the charge conjugation matrices of so(4,1) and so(5) used to raise and lower spinor indices. Note that $C \times C'$ is symmetric, i.e. $\bar{\theta}\theta = 0$. In the expressions below θ^I may be thought of as 5-d spinors with an extra 'spectator' 5-d spinor index, and we shall assume that γ^a and γ^p stand for $\gamma^a \times I_4$ and $I_4 \times \gamma^p$. $C\gamma^{a_1\cdots a_n}$ are symmetric (antisymmetric) for $n = 2,3$ mod 4 ($n = 0,1$ mod 4). The same properties are valid for $C'\gamma^{p_1\cdots p_n}$. We also assume that $\gamma_0^\dagger = -\gamma_0$, $\gamma_m^\dagger = \gamma_m$ $(m = 1,2,3,4)$.

background [5, 26]. To check this we treat g_{ij} as an arbitrary background metric, *not* identifying it (both in the action and in the path integral measure) with h_{ij}.

Let us introduce the tangent-space components of the fluctuation fields which have the canonical norms

$$\zeta^a = E^a_\mu \xi^\mu\,, \qquad \zeta^p = E^p_\mu \xi^\mu\,, \qquad G_{\mu\nu} = E^a_\mu E^a_\nu + E^p_\mu E^p_\nu\,,$$
$$\|\zeta^a\|^2 = \int d^2\sigma\sqrt{g}\,\zeta^a\zeta^a\,, \qquad \|\zeta^p\|^2 = \int d^2\sigma\sqrt{g}\,\zeta^p\zeta^p\,. \tag{3.10}$$

ζ^a and ζ^q are the fluctuations of the AdS_5 and S^5 coordinates respectively (the tangent-space 5-d indices $a, b = 0, \ldots, 4$ and $p, q = 1, \ldots, 5$ are raised by the flat 5-d metrics). We then get the following action for the quadratic fluctuations (we absorb $1/2\pi\alpha'$ by rescaling the quantum fields)

$$S_{2\mathrm{B}} = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}D_i\zeta^a D_j\zeta^a + X_{ab}\zeta^a\zeta^b + g^{ij}D_i\zeta^q D_j\zeta^q + X_{pq}\zeta^p\zeta^q\right), \tag{3.11}$$
$$X_{ab} = -g^{ij}\eta^c_i\eta^d_j R_{acbd}\,, \qquad X_{pq} = -g^{ij}\eta^r_i\eta^s_j R_{prqs}\,. \tag{3.12}$$

Here

$$\eta^a_i \equiv \partial_i\bar{x}^\mu E^a_\mu\,, \qquad \eta^p_i \equiv \partial_i\bar{x}^\mu E^p_\mu\,, \tag{3.13}$$

are the projection of the AdS_5 and S^5 vielbeins on the world sheet. D_i is the covariant derivative containing the projection of the target space spin connection,

$$D_i\zeta^a = \partial_i\zeta^a + w^{ab}_i\zeta^b\,, \qquad w^{ab}_i = \partial_i\bar{x}^\mu\Omega^{ab}_\mu\,, \tag{3.14}$$

where Ω^{ab}_μ is the spin connection of AdS_5, and similarly for S^5. For example, the AdS_5 part of the connection is $\Omega^{m4}_\mu = -w^{-1}\delta^m_\mu$, where 4 stands for the radial direction and $m = 0, 1, 2, 3$.

In general, there will be two types of divergences — depending on the background $\bar{x}$ field $O(\partial\bar{x}\partial\bar{x})$ (i.e. renormalization of the target space metric) and proportional to the curvature $R^{(2)}$ of the fiducial 2-d metric g_{ij} (i.e. renormalization of the dilaton).

To check the conformal invariance we need to use the fact that for $AdS_5 \times S^5$

$$R_{acbd} = -\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{cb}\,, \qquad R_{prqs} = \delta_{pq}\delta_{rs} - \delta_{ps}\delta_{rq}\,. \tag{3.15}$$

Then

$$X^{ab} = g^{ij}\eta^c_i\eta^c_j\delta^{ab} - g^{ij}\eta^a_i\eta^b_j, \qquad X^{pq} = -g^{ij}\eta^r_i\eta^r_j\delta^{pq} + g^{ij}\eta^p_i\eta^q_j\,. \tag{3.16}$$

The $\bar{x}$-dependent logarithmic UV divergences coming from (3.11) are proportional to

$$\mathrm{tr}\, X = 4g^{ij}\left(\eta^a_i\eta^a_j - \eta^p_i\eta^p_j\right). \tag{3.17}$$

This gives the Ricci tensor dependent of the conformal invariance equation, $R_{\hat{a}\hat{b}} - \frac{1}{4\cdot 4!}F_{\hat{a}\ldots.}F_{\hat{b}\ldots.} = 0$. The 5-form dependent part will come from the fermionic contribution.

κ-symmetry transformations in curved $AdS_5 \times S^5$ space [5] which leave (3.4) invariant have a form similar to flat-space transformations (2.6)

$$\delta_\kappa \theta^I = \tilde{\rho}_i^\dagger \kappa^{iI} + \cdots \,, \tag{3.18}$$

where

$$\frac{1}{\sqrt{g}} \epsilon^{ij} \kappa_j^1 = -\kappa^{i1} \,, \qquad \frac{1}{\sqrt{g}} \epsilon^{ij} \kappa_j^2 = \kappa^{i2} \,, \tag{3.19}$$

and (cf. (3.7))

$$\tilde{\rho}_i^\dagger = \left(\Gamma_a E_\mu^a - i \Gamma_p E_\mu^p\right) \partial_i x^\mu \,. \tag{3.20}$$

Fixing the κ-symmetry gauge

$$\theta^1 = \theta^2 = \theta \,, \tag{3.21}$$

the quadratic part of the fermionic action (3.4)–(3.7) is found to be

$$S_{2\mathrm{F}} = 2i \int d^2\sigma \left(\sqrt{g} g^{ij} \bar{\theta} \rho_i \mathcal{D}_j \theta - \frac{i}{2} \epsilon^{kj} \bar{\theta} \rho_k \tilde{\rho}_j \theta \right) . \tag{3.22}$$

We shall first be interested in $\partial\bar{x}\partial\bar{x}$ divergences so that it should not be necessary to distinguish between g_{ij} and the induced metric, therefore

$$\rho_{(i}\rho_{j)} = g_{ij} = G_{\mu\nu} \partial_i \bar{x}^\mu \partial_j \bar{x}^\nu = \eta_i^a \eta_j^a + \eta_i^p \eta_j^p \,. \tag{3.23}$$

In general, given the operator $\mathcal{O} = i\rho^k \mathcal{D}_k + M$, the logarithmic divergence in $\frac{1}{2} \ln \det(\mathcal{O}\mathcal{O}^\dagger)$ is proportional to[5]

$$-\frac{1}{4} g^{ij} \operatorname{tr}(\rho_i M \rho_j M + \rho_i M^\dagger \rho_j M^\dagger) \,, \tag{3.24}$$

where in the present case $M = \frac{1}{2} \frac{\epsilon^{kj}}{\sqrt{g}} \rho_k \tilde{\rho}_j$, $M^\dagger = -\frac{1}{2} \frac{\epsilon^{kj}}{\sqrt{g}} \tilde{\rho}_k^\dagger \rho_j$, $\rho_i = \rho_i^\dagger$. After some algebra[6] one finds that (3.24) reduces to

$$-4 \left[\det\left(\eta_i^a \eta_j^a\right) - \det\left(\eta_i^p \eta_j^p\right)\right] = 4 - g^{ij} \left(\eta_i^a \eta_j^a - \eta_i^p \eta_j^p\right) . \tag{3.25}$$

This combination determines the fermionic contribution to the $\bar{x}$-dependent logarithmic divergence, and it exactly *cancels* the bosonic contribution (3.17).

As usual, fixing conformal gauge produces bosonic ghosts which are 2-d vectors

$$S_{\mathrm{gh}} = \frac{1}{2} \int d^2\sigma \sqrt{g} g^{ij} \left(g^{kl} \nabla_k \epsilon_i \nabla_l \epsilon_j - \frac{1}{2} R^{(2)} \epsilon_i \epsilon_j \right) , \tag{3.26}$$

where $R^{(2)}$ is the scalar curvature of the 2-d metric g_{ij}.

[5] Starting with $(i\rho^k \partial_k + M)(i\rho^n \partial_n + M^\dagger)$ one is to note that in the general case of $\rho^k M^\dagger + M\rho^k \neq 0$ one is to introduce an additional connection to put the resulting operator in the standard form $-D^2 + X$.

[6] If the background does not depend on S^5, then $\rho_i = \tilde{\rho}_i$ and the calculation is trivial.

The quadratic and linear divergences cancel between bosons and fermions because of the matching of the number of degrees of freedom. The coefficients of the logarithmic divergence for the above system of fields (the Seeley coefficients of the corresponding second order Laplace operators) are

$$b_{2\rm B} = 10 \times \frac{R^{(2)}}{6} - {\rm tr}\, X \,, \qquad b_{2\rm gh} = -2 \times \frac{R^{(2)}}{6} - R^{(2)}\,,$$
$$b_{2\rm F} = 8 \times \frac{R^{(2)}}{3} + {\rm tr}\, X \,. \tag{3.27}$$

Here we took into account that since the kinetic part of the fermionic operator depends on the background 2-d metric through $\sqrt{g}g^{ij}$, the $R^{(2)}$-dependent part of its divergence and conformal anomaly coefficient is four times greater than for a 2-d Majorana fermion, just like in the flat GS string case (2.8) (this difference may be attributed to the contribution of the jacobian of a local rotation that transforms ρ_i into 2-d Dirac matrices contracted with zweibein, see appendix C).

The total divergence coefficient is then

$$b_2^{\rm total} = (10 - 2 + 8 \times 2) \times \frac{R^{(2)}}{6} - R^{(2)} = 3 R^{(2)}\,. \tag{3.28}$$

As was mentioned above, at 1-loop order the argument for the cancellation of logarithmic divergences is identical to the argument in the case of the flat GS string, where (3.28) is also valid. Integrating over the scalar curvature on a closed surface will give the Euler character $\int d^2\sigma \sqrt{g}\, 3R^{(2)} = 12\pi\chi$. The same is true on a surface with boundary where, as was shown in [12], all the factors of $R^{(2)}$ are accompanied by the appropriate boundary term. Now one should remember that the cutoff dependent factors in the conformal Killing vector and/or Teichmüller measure exactly cancels this divergence, so the final result is $(D-10)\chi$, namely zero. This is, of course, consistent with the cancellation of the total $\bar{x}$-independent conformal anomaly, or the central charge, ensuring that the dilaton equation is satisfied. Note that this is just a consequence of working in the critical dimension.

Thus, we have confirmed that the theory is conformal at one loop. As was argued in [5, 26], this should be true to all orders in α' expansion (for example, the first non-trivial correction to the central charge vanishes because the Ricci scalar of the target space metric is zero, $R_{tot} = R_{AdS_5} + R_{S^5} = 0$, etc.).

3.3 Nambu-type action in the static gauge

Alternatively, one may start with the corresponding Nambu-Goto form of the GS action (with no independent 2-d metric). This action is highly non-linear, but in the quadratic approximation it is straightforward to determine the 2-d operators of small fluctuations of a string in curved background. Here it is natural to choose the static gauge to fix the diffeomorphisms, i.e. to identify the world sheet coordinates

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with the two target space coordinates and demand that there are no fluctuations in those directions. The ghost determinant is then "local", i.e. is a determinant of an operator of multiplication by a function (which needs a regularization and may still produce non-trivial contribution to partition function).

Following the standard procedure (for a careful treatment see, e.g., [27]), fixing the static gauge $\delta x^k = 0$, $k = 0, 1$, produces the ghost determinant

$$\Delta_{\rm gh}^{-1} = \int [d\epsilon] \exp\left(-\frac{1}{2}||\delta_\epsilon x^k||^2\right) . \tag{3.29}$$

The path integral over the 2-d diffeomorphism parameters ϵ^i is defined using the norm

$$||\epsilon||^2 = \int d^2\sigma \sqrt{h} h_{ij} \epsilon^i \epsilon^j . \tag{3.30}$$

Here h_{ij} is the induced metric (3.8)

$$h_{ij} = G_{\mu\nu}(\bar{x}) \partial_i \bar{x}^\mu \partial_j \bar{x}^\nu = \eta_i^{\hat{a}} \eta_j^{\hat{a}} . \tag{3.31}$$

Explicit evaluation of (3.29) gives

$$\begin{aligned} ||\delta_\epsilon x^k||^2 &= \int d^2\sigma \sqrt{h} G_{kl}(\bar{x})(\epsilon^i \partial_i \bar{x}^k)(\epsilon^j \partial_j \bar{x}^l) = \int d^2\sigma \sqrt{h} h^{||}_{ij} \epsilon^i \epsilon^j , \\ h^{||}_{ij} &\equiv G_{kl}(\bar{x}) \partial_i \bar{x}^k \partial_j \bar{x}^l , \end{aligned} \tag{3.32}$$

so that

$$\Delta_{\rm gh} = [\det(h^{||}_{ik} h^{kj})]^{1/2} . \tag{3.33}$$

As we will see on the examples discussed below, the most natural regularization of this "local" determinant is by changing the normalization of some of the fluctuating fields.

Another possible gauge is to remove the vielbein components of the longitudinal fluctuations. It is easy to see, by the same calculation, that the ghost determinant is equal to 1 in that gauge.

The resulting bosonic action is a modification of (3.11)

$$S = \frac{1}{2} \int d^2\sigma \sqrt{h} \left(h^{ij} D_i \zeta^{\bar{a}} D_j \zeta^{\bar{a}} + \bar{X}_{\bar{a}\bar{b}} \zeta^{\bar{a}} \zeta^{\bar{b}}\right) , \tag{3.34}$$

where $\zeta^{\bar{a}}$ are the fields representing the transverse fluctuations. $\bar{X}$ is not the same as X (in the simple examples discussed below $\mathrm{tr}\, \bar{X} = \mathrm{tr}\, X + R^{(2)}$). The fermions are treated in the same way as before, so squaring the fermionic operator gives a mass term whose trace is equal to $\mathrm{tr}\, X$.

The non-trivial $O(\partial\bar{x}\partial\bar{x})$ part of divergences cancels again, while the remaining $\int R^{(2)}(h)$ part (which, in the presence of a boundary should be accompanied by an appropriate boundary term to give the Euler number) should be canceled in $D = 10$ by appropriate measure factor contributions, as happens in conformal gauge [12].

It should be stressed that, while the result of a semiclassical computation in the Nambu action case should be equivalent to the one in the Polyakov action case [28], a careful definition of the path integration measure is non-trivial in the Nambu case. For that reason we prefer to use the Polyakov definition of the string partition function which is well-defined. It should be clear that the problem of cancellation of $\int R^{(2)}$ divergences is exactly the same as in the case of the Nambu action in flat space, and thus has nothing to do with peculiarities of the $AdS_5 \times S^5$ background. This resolves the puzzle of the apparent non-cancellation of logarithmic divergences that was encountered in [7], which is revealed to be an artifact of the use of the Nambu-type formulation without including the additional measure contributions to the divergences and ignoring the subtleties of the divergences/conformal anomaly cancellation in the flat-space GS superstring theory.

We shall see that in the cases of interest the kinetic operator of GS fermions takes (after a local rotation) the form of the 2-d Dirac operator in the curved 2-d geometry defined by the induced metric. In view of the above discussion, the fact that when one directly evaluates the divergences one seems to find that the $R^{(2)}$-terms do not cancel is an artifact of not distinguishing between the generic and the induced metrics. These topological divergences are, in any case, irrelevant for the evaluation of the non-trivial part of the partition function which determines the correction to the Wilson loop expectation value, or to the $1/L$ potential. The issue of divergences may be avoided altogether, by normalizing, as we suggest below, the partition function to its value for some standard background.

Once this issue has been clarified, one should be able to use the static gauge expressions for the small-fluctuation determinants, as they sometimes turn out to be simpler than the analogous expressions in the Polyakov formulation in conformal gauge. In what follows we shall not distinguish between the generic fiducial metric g_{ij} and the induced metric h_{ij} (using always the notation g_{ij} for the 2-d metric).

3.4 Relating quadratic GS fermion term to 2-d Dirac fermion action

Before turning to specific examples let us make some general comments on how one can put the quadratic fermionic term (3.22) in the GS action in curved target space background into the standard kinetic term for a set of 2-d fermions defined on a curved 2-d space. The main idea is to apply a local target space Lorentz rotation to GS spinor θ, as discussed previously in the case of the heterotic string in flat [13, 14, 15] and curved [16, 17] spaces. We shall concentrate on the derivative term in the fermionic action (3.22). It should be noted that the presence of the second "mass" term in (3.22) which originates from the coupling to RR background and which is absent in the heterotic case will not allow to compute the resulting 2-d fermion determinant in a closed form using the standard anomaly [29] arguments.

Ignoring the distinction between g_{ij} and the induced metric (3.31) we can write the derivative term in (3.22) as

$$S_{2\mathrm{F}}^{(deriv.)} = 2i \int d^2\sigma \sqrt{g} g^{ij} \bar{\theta} \rho_i \mathcal{D}_j \theta = i \int d^2\sigma \sqrt{g} g^{ij} \left(\bar{\theta} \rho_i \partial_j \theta - \partial_j \bar{\theta} \rho_i \theta \right). \qquad (3.35)$$

Let us introduce the tangent t^μ_α ($\mu = 0, 1, \ldots, 9$, $\alpha = 0, 1$) and normal n^μ_s ($s = 1, \ldots, 8$) vectors to the world surface which form orthonormal 10-d basis ($g_{ij} = e^\alpha_i e^\beta_j \eta_{\alpha\beta}$)

$$t^\mu_\alpha = e^i_\alpha \partial_i \bar{x}^\mu , \qquad (t_\alpha, t_\beta) = \eta_{\alpha\beta} , \qquad (t_\alpha, n_s) = 0 , \qquad (n_s, n_u) = \delta_{su} , \qquad (3.36)$$

where $(a, b) = G_{\mu\nu} a^\mu b^\nu$. Then one can make a local SO(1, 9) rotation of this basis which transforms the set of σ-dependent 10-d Dirac matrices (see (3.5)) into the 10 constant Dirac matrices

$$\rho_\alpha(\sigma) = e^i_\alpha \rho_i = S(\sigma) \Gamma_\alpha S^{-1}(\sigma) , \qquad \rho_s(\sigma) = n^\mu_s E^{\hat{a}}_\mu \Gamma_{\hat{a}} = S(\sigma) \Gamma_s S^{-1}(\sigma) . \qquad (3.37)$$

One may further choose a representation in which $\Gamma_\alpha = \tau_\alpha \times I_8$, where τ_α are 2-d Dirac matrices. Depending on the specific embedding and particular curved target space metric, one may then be able to write the action (3.35) as the action for 2-d Dirac fermions coupled to curved induced 2-d metric and interacting with some gauge fields (coming from $S^{-1} dS$).

Simple examples when this happens will be discussed below. We shall consider embeddings of the string world sheet into the AdS_3 part of the AdS_5 space, so there will be only *one* normal direction and the extra normal bundle 2-d gauge connection will be absent (cf. [13, 14, 16]). In this 3-dimensional embedding case with non-chiral 2-d fermions the jacobian associated with the local Lorentz rotation will be trivial (see also appendix C).

4. One-loop approximation near the straight string configuration: supersymmetric field theory on AdS_2

The simplest classical solution for string in $AdS_5 \times S^5$ is a straight string with the world surface spanned by the radial direction of AdS_5 and time. The euclidean solution and the corresponding induced metric are

$$\tau = x^0 , \qquad \sigma = x^4 = w , \qquad ds^2 = \frac{1}{\sigma^2}(d\tau^2 + d\sigma^2) . \qquad (4.1)$$

The induced metric on the world sheet is that of AdS_2, with constant negative curvature $R^{(2)} = -2$ (the radius of AdS_5 is $R = 1$).

This solution represents a single straight Wilson line running along the euclidean time direction. This is a BPS object in string theory, it corresponds to a static fundamental string stretched between a single D3-brane and N coinciding D3-branes. Therefore, one would expect that the partition function be equal to 1. The properly defined (subtracted) classical string action evaluated on (4.1) indeed vanishes, and we shall evaluate the 1-loop correction to the partition function.

As we shall show, the corresponding 1-loop correction to the vacuum energy defined with respect to a certain time-like Killing vector vanishes. Relating the vacuum energy to the partition function using a conformal rescaling argument (and the fact that the total conformal anomaly is zero) we conclude that $Z = 1$. It should be mentioned that while the (properly defined) vacuum energy of a supersymmetric field theory in AdS space should vanish, this does *not* automatically imply (in contrast to what happens in flat space) that the partition function of such theory should be equal to 1 (cf. [30, 31]). In the present case this happens only with the inclusion of the appropriate ghosts and longitudinal modes. The calculation of the partition function is rather subtle, and depends on a regularization prescription. Let us note also that the point of view of physical applications, the precise value of Z (which is simply a constant) is not actually important, and one may normalize with respect to it in computing Z for more general string configurations.

Apart from being the simplest example, there are other reasons why the analysis of the straight string case is of interest. Any smooth Wilson loop looks in the UV region like a straight line. In the present set-up this translates into the behavior of the minimal surface near the boundary of AdS_5 space. In the general case one will have to calculate the partition function for a complicated two dimensional field theory. But asymptotically the minimal surface will approach AdS_2, and the small fluctuation operators (in particular, the asymptotic values of the masses of the fluctuation fields) will also be the same as for a straight string. Many subtleties related to divergences and asymptotic boundary conditions are already present in this example, and they can be automatically avoided in more general cases by normalizing with respect to the partition function of the straight string.

4.1 The action and multiplet structure

The bosonic part of the action for small fluctuations in conformal gauge is (3.11)

$$S_{2\mathrm{B}} = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}D_i\zeta^a D_j\zeta^a + X_{ab}\zeta^a\zeta^b + g^{ij}D_i\zeta^p D_j\zeta^p\right), \tag{4.2}$$

where in the present case

$$\eta_0^a = \partial_0\bar{x}^\mu E_\mu^a = w^{-1}(1,0,0,0,0)\,, \qquad \eta_1^a = \partial_1\bar{x}^\mu E_\mu^a = w^{-1}(0,0,0,0,1)\,. \tag{4.3}$$

η_i^a with $a = 0, 4$ is thus just a vielbein of the induced 2-d metric g_{ij},

$$\eta_i^a = e_i^{\ \alpha}\,, \qquad a = 0,4\,, \quad \alpha = 0,1\,, \tag{4.4}$$

so that

$$X_{ab} = \mathrm{diag}(1, 2, 2, 2, 1)\,. \tag{4.5}$$

The only nonzero connection in D_i (3.14) is $w_0^{04} = -w^{-1}$, i.e.

$$D_0\zeta^0 = \partial_0\zeta^0 - w^{-1}\zeta^4, \qquad D_0\zeta^4 = \partial_0\zeta^4 + w^{-1}\zeta^0. \tag{4.6}$$

The natural norms for the fields are

$$||\zeta^{\hat{a}}||^2 = \int d^2\sigma\sqrt{g}\zeta^{\hat{a}}\zeta^{\hat{a}} = \int d\tau d\sigma\frac{1}{\sigma^2}\zeta^{\hat{a}}\zeta^{\hat{a}}\,. \tag{4.7}$$

The ghost action is the same as in (3.26), i.e.

$$\frac{1}{2}\int d^2\sigma\sqrt{g}\left(\nabla^i\epsilon^\alpha\nabla_i\epsilon^\alpha - \frac{1}{2}R^{(2)}\epsilon^\alpha\epsilon^\alpha\right), \tag{4.8}$$

where we defined $\epsilon^\alpha = e_i^\alpha\epsilon^i$ with flat 2-d tangent space indices, and ∇_i includes the world sheet Lorentz connection.

Because of the direct embedding of the world sheet into the target space there are some extra simplifications. The projection of the target space connection on the world sheet w_i^{ab} is the same as the spin connection of the induced metric appearing in ∇_i. In addition, $-\frac{1}{2}R^{(2)} = 1$. Therefore, the action of the ghosts is identical to the action of the longitudinal modes ζ^0, ζ^4, but the boundary conditions are different [12].

Before κ-symmetry gauge fixing the fermionic lagrangian (3.4) is (here we use Minkowski notation)

$$L_{2\mathrm{F}} = -i\left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}s^{IJ}\right)\bar{\theta}^I\rho_i D_j\theta^J\,, \tag{4.9}$$

where in the present case

$$\begin{aligned}
\rho_i &= e_i^\alpha\rho_\alpha = \eta_i^{\ a}\Gamma_a = \begin{cases} w^{-1}\Gamma_0 & \text{for } i = 0\,, \\ w^{-1}\Gamma_4 & \text{for } i = 1\,, \end{cases} \\
D_i\theta^J &= \hat{\nabla}_i\theta^J - \frac{i}{2}\epsilon^{JK}\rho_i\theta^K\,, \\
\hat{\nabla}_0 &= \partial_0 - \frac{1}{2w}\Gamma_{04}\,, \qquad \hat{\nabla}_1 = \partial_1\,.
\end{aligned} \tag{4.10}$$

We choose again the gauge $\theta^1 = \theta^2 = \theta$. Then

$$L_{2\mathrm{F}} = -2i\sqrt{g}\left(\bar{\theta}\rho^i\hat{\nabla}_i\theta + i\bar{\theta}\rho_3\theta\right), \qquad \rho_3 \equiv \frac{1}{2}\epsilon^{\alpha\beta}\rho_\alpha\rho_\beta = \Gamma_{04}\,. \tag{4.11}$$

Here we introduced the notation $\rho_\alpha = (\Gamma_0, \Gamma_4)$ (ρ_α may be identified with 2-d Dirac matrices times I_8). Thus the quadratic fermionic part of GS action has exactly the same form as the action for 2-d fermions in curved 2-d space.

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Assuming the standard $\int d^2\sigma\sqrt{g}\bar{\theta}\theta$ normalization, the corresponding Dirac operator is

$$D_F = i\rho^i\hat{\nabla}_i - \rho_3 = iw(-\Gamma_0\partial_0 + \Gamma_4\partial_1) - \frac{1}{2}i\Gamma_4 - \Gamma_0\Gamma_4\,, \tag{4.12}$$

where the third term came from D_0. The spectral problem is thus

$$\left[iw(-\Gamma_0\partial_0 + \Gamma_4\partial_1) - \frac{1}{2}i\Gamma_4 - \Gamma_0\Gamma_4\right]\theta = \lambda\theta\,. \tag{4.13}$$

Directly squaring this operator we get

$$\left(w^2\left[(\partial_0 - \frac{1}{2}w^{-1}\Gamma_0\Gamma_4)^2 - \partial_1^2\right] + \frac{1}{2}\right)\theta = \left(-\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1\right)\theta = \lambda^2\theta\,. \tag{4.14}$$

Ignoring the ghosts and longitudinal modes, we are left with a 2-d field theory on AdS_2 containing five massless scalars, three scalars with mass squared 2, and eight fermions with mass squared 1. Field theories on AdS_2 were studied in the past (see [32, 33, 34, 35, 36, 30, 37]). The fields in the $\mathcal{N}=1$ scalar supermultiplet in AdS_2 may have the following bosonic and fermionic masses [38, 30, 33]:

$$m_B^2 = \mu^2 - \mu\,, \qquad m_F = \mu\,, \tag{4.15}$$

where μ is a free parameter. In the case at hand we have 5 "massless" multiplets with $\mu = 1$ ($m_B^2 = 0$, $m_F = 1$) and 3 "massive" multiplets with $\mu = -1$ ($m_B^2 = 2$, $m_F = -1$). It is possible to combine a $\mu = 1$ and a $\mu = -1$ multiplet into an $\mathcal{N} = 2$ multiplet, the dimensional reduction of the 4-d chiral multiplet to 2 dimensions. Two $\mu = 1$ multiplets also form an $\mathcal{N} = 2$ multiplet which is the dimensional reduction of the 4-d vector multiplet. Three chiral and one vector multiplets in $D = 4$ make one $\mathcal{N} = 4$ vector in four dimensions, so we conclude that the 8 scalars and 8 fermions that we have found should form one $\mathcal{N} = 8$ multiplet in two dimensions (see, e.g., [39] for related discussion).[7]

We finally obtain the following partition function with the scalar and spinor Laplace operators defined with respect to the euclidean AdS_2 metric with radius 1 ($R^{(2)} = -2$)

$$Z_{B+F} = \frac{\det^{8/2}\left(-\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1\right)}{\det^{3/2}(-\nabla^2 + 2)\det^{5/2}(-\nabla^2)}\,. \tag{4.16}$$

It seems reasonable to impose, as is usually done in discussions of supersymmetric theories in AdS_n backgrounds [40, 41], proper boundary conditions consistent with

[7] The mass term in the action (4.11), contains the matrix $\rho_3 = \Gamma_0\Gamma_4$ which has half of its eigenvalues 1 and half -1, i.e. there are actually 4 fermions with $m_F = -1$ and 4 with $m_F = 1$, not $3+5$. But the sign choice in (4.15) $m_B^2 = \mu^2 - \mu$ rather than $m_B^2 = \mu^2 + \mu$ is for $\mathcal{N} = 1$ supersymmetry. For extended supersymmetry both signs are possible, so the bosons can be split into 3 and 5 while the fermions are split to 4 and 4.

supersymmetry. Those imply that the resulting spectra of Laplace operators are discrete in spatial direction (and not continuous as one would normally expect to find in a non-compact hyperbolic space).

A direct calculation of the partition function would involve solving the spectral problems (4.14) and

$$\left(-\partial_0^2 - \partial_1^2 + \frac{m^2}{\sigma^2}\right)\zeta = \frac{\lambda}{\sigma^2}\zeta\,, \tag{4.17}$$

with $m^2 = 0, 2$. The solutions to the bosonic problem which vanish at $\sigma = 0$ are

$$\zeta(\tau,\sigma) = e^{ip\tau}\sqrt{\sigma}K_{i\nu}(p\sigma)\,, \qquad \nu^2 = \lambda - m^2 - \frac{1}{4}\,, \qquad 0 \le \nu < \infty\,, \tag{4.18}$$

where $K_{i\nu}$ are modified Bessel functions.

A calculation of the partition function based on this spectrum is presented in appendix B. This direct approach suffers from regularization problems, it also does not capture the symmetries of the problem, like supersymmetry. Here we use a different method to evaluate it.

4.2 Vacuum energy

Instead of calculating the partition function directly, we could start with the vacuum energy. It is given by the determinant of the operator scaled to remove the factor of g^{00} from in front of ∂_t^2 (see [31]). Then we use the conformal anomaly, as discussed in appendix A, to derive the partition function.

Let us change the world sheet coordinates so that the AdS_2 metric is

$$ds^2 = \frac{1}{\cos^2\rho}\left(dt^2 + d\rho^2\right)\,, \qquad \rho \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{4.19}$$

The spectra of the hamiltonians conjugate to this time variable were calculated in [34]:[8]

$$\omega_n^{(F)}(\mu) = n + |\mu| + \frac{1}{2}\,, \qquad \omega_n^{(F)}(\pm 1) = n + \frac{3}{2}\,, \tag{4.20}$$
$$\omega_n^{(B)}(\mu) = n + h(\mu)\,, \quad h(\mu) = \frac{1}{2}\left(1 + \sqrt{1 + 4m_B^2}\right)\,, \qquad h(-1) = 2\,, \quad h(1) = 1\,.$$

Summing over all the modes we get, as in [33], the 1-loop vacuum energy of this effective 2-d field theory

$$E = \frac{1}{2}\sum_{n=0}^{\infty}\left(3\left[\omega_n^{(B)}(-1) - \omega_n^{(F)}(-1)\right] + 5\left[\omega_n^{(B)}(1) - \omega_n^{(F)}(1)\right]\right). \tag{4.21}$$

[8]From the group-theoretical point of view, the unitary irreducible representations of the AdS_2 superalgebra contain: for $\mu > 1/2$ a scalar field with $\omega_n^{(B)} = n + \mu$ and a fermion field with $\omega_n^{(F)} = n + \mu + 1/2$, and for $\mu < -1/2$ — a scalar field with $\omega_n^{(B)} = n + 2|\mu|$ and a fermion field with $\omega_n^{(F)} = n + |\mu| + 1/2$.

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As was extensively discussed in the literature, the *properly defined* vacuum energy should *vanish* in the AdS case as it does in flat space [32, 34, 30, 42, 43] (even though divergences may not cancel out, unless there is a lot of supersymmetry [44]). However, the direct computation of the sum of the mode energies using ζ-function regularization may lead to a non-zero result because the ζ-function regularization may not, in general, preserve supersymmetry.

Using the standard relations

$$\zeta(s,x) \equiv \sum_{n=0}^{\infty}(n+x)^{-s}\,, \qquad \zeta(-1,x) = -\frac{1}{2}\left(x^2 - x + \frac{1}{6}\right), \tag{4.22}$$

we find for a boson ($m_B^2 = \mu^2 - \mu$)

$$E_B = \frac{1}{2}\sum_{n=0}^{\infty}[n + h(\mu)] = \frac{1}{2}\zeta(-1, h(\mu)) = -\frac{1}{4}\left(m_B^2 + \frac{1}{6}\right), \tag{4.23}$$

and for a fermion ($m_F = \mu$)

$$E_F = -\frac{1}{2}\sum_{n=0}^{\infty}\left(n + |\mu| + \frac{1}{2}\right) = -\frac{1}{2}\zeta\left(-1, |\mu| + \frac{1}{2}\right) = \frac{1}{4}\left(m_F^2 - \frac{1}{12}\right). \tag{4.24}$$

In our case we find[9]

$$E = -\frac{1}{4}\left[3 \times \left(2 + \frac{1}{6}\right) + 5 \times \frac{1}{6} - 8 \times \left(1 - \frac{1}{12}\right)\right] = 0\,. \tag{4.25}$$

The ratio 3 : 5 of the numbers of the two multiplets is just what is needed for the cancellation.

The fact that E, defined by ζ-function regularization, vanishes may be a consequence of the extended $\mathcal{N} = 8$ supersymmetry mentioned above. Indeed, while as was originally suggested [32] for AdS_4 and confirmed also in [30, 42, 43], the ζ-function regularization may break supersymmetry and thus may lead to $E \neq 0$, this does not actually happen in the case of $N \geq 5$, $D = 4$ gauged supergravities [44]. The present $D = 2$ case is thus analogous to those $D = 4$ cases with large amounts of supersymmetry.[10]

[9]Note that if we set the mass terms to zero by taking $\mu = 0$ in (4.15), we get 8 massless scalars and 8 massless fermions in AdS_2 and their vacuum energies do not cancel — we get $\frac{1}{2} \times 8[\zeta(-1,1) - \zeta(-1,1/2)] = 4 \times (-1/8)$.

[10]The $E = 0$ property of $N \geq 5$, $D = 4$ supergravities might be related to the cancellation [45] of the logarithmic gauge coupling renormalization in these theories (note, in particular, that as was discussed in [46] in the case of the flat space, the vacuum energy as defined by the partition function is the same as the sum of zero-point energies provided $\sum \zeta(0) = 0$, i.e. if there are no UV infinities). It may seem that the analogy between our $D = 2$ case and the $D = 4$ cases is not quite complete since here, in fact, the naive calculation of the coefficient of the logarithmic divergence in terms of the sum of ζ-functions gives a nonzero answer (using $\zeta(0,x) = 1/2 - x$, we find the total coefficient to be 1). Note, however, that the types of divergences which cancel in the $D = 4$ cases and do not cancel in the $D = 2$ case are actually quite different, i.e. the direct comparison is not possible.

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4.3 Partition function

The vacuum energy calculated in the previous subsection as the sum over oscillator modes corresponds to the determinants of the following (mass m) bosonic and fermionic spectral problems

$$\left(-\partial_0^2 - \partial_1^2 + \frac{m^2}{\cos^2\rho}\right)\zeta = \lambda_B \zeta\,,$$
$$\left(-\hat{\nabla}_0^2 - \hat{\nabla}_1^2 + \frac{1}{2}\frac{1}{\cos^2\rho}\right)\theta = \lambda_F^2 \theta\,, \qquad (4.26)$$

where in the fermionic operator we assume that the covariant derivatives are contracted using flat metric. These are related to the spectral problems (4.14) and (4.17) (apart from the coordinate change) by a rescaling of the right hand side by $\cos^2\rho$. As was mentioned above, in curved (e.g., static conformally flat) space the logarithm of the partition function is, in general, different from the vacuum energy defined as a sum over eigen-modes because the time derivative part of the relevant elliptic operators is rescaled by g^{00}. The determinants of the two operators which differ by such a rescaling are related to each other by a conformal anomaly type equation as discussed in appendix A. The extra contribution from a mass m boson is

$$-\log\det\Delta_1 = -\log\det\Delta_M + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(m^2\ln M + \frac{1}{12}\partial^i\ln M\partial_i\ln M\right), \quad (4.27)$$

where $M = \cos^2\rho = 1/\sqrt{g}$. The two terms in the parentheses differ only by a total derivative, but we choose to write it this way to eliminate the boundary terms. Each fermion contributes

$$\log\det\Delta_1 = \log\det\Delta_{\mathcal{K}^2} - \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(2\ln\mathcal{K} - \frac{2}{3}\partial^i\ln\mathcal{K}\,\partial_i\ln\mathcal{K}\right), \qquad (4.28)$$

where $\mathcal{K} = \cos\rho = \sqrt{M}$. Summed together the 'transverse' scalars and fermions contribute

$$\log Z_{B+F} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(\ln M - \partial^i\ln M\partial_i\ln M\right). \qquad (4.29)$$

To this we should add the contribution of the ghosts and longitudinal modes. For the ghosts one gets the standard Liouville action

$$\log\det\Delta_1^{\rm gh} = \log\det\Delta_M^{\rm gh} - \frac{26}{12}\times\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\partial^i\ln M\partial_i\ln M\,. \qquad (4.30)$$

The longitudinal modes have the same action as the ghosts, but different boundary conditions giving the conformal anomaly

$$-\log\det\Delta_1^L = -\log\det\Delta_M^L + 2\times\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(\ln M + \frac{1}{12}\partial^i\ln M\partial_i\ln M\right), \quad (4.31)$$

so that

$$\log Z_{L+{\rm gh}} = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(\ln M - \partial^i\ln M\partial_i\ln M\right). \qquad (4.32)$$

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Putting it all together we find that the partition function is identically equal to one

$$Z_{\text{total}} = \frac{\det^{1/2}(-\Delta^{\text{gh}}_{ij} + \delta_{ij}) \det^{8/2}(-\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1)}{\det^{1/2}(-\Delta_{ij} + \delta_{ij}) \det^{3/2}(-\nabla^2 + 2) \det^{5/2}(-\nabla^2)} = 1\,. \tag{4.33}$$

This result is a consequence of the fact that the vacuum energy vanishes, and also of the cancellation of the sum of conformal anomalies for ten bosons with total mass terms 8, eight fermions with 4 times the standard 2-d fermion conformal anomaly and total mass 8, and the conformal gauge ghosts.

Note that this is not identical to the conformal anomaly calculation of section 3.2. Here we did *not* distinguish between the induced and the fiducial metric. An alternative method of calculating the partition function would be to go back and treat the fiducial metric g_{ij} and the induced metric h_{ij} as independent. Then only the fiducial metric should be rescaled, while the induced metric should not. It is most convenient to work with flat metric on the strip

$$g_{ij} = \delta_{ij}\,, \qquad h_{ij} = \frac{1}{\cos^2\rho}\delta_{ij}\,. \tag{4.34}$$

That eliminates the problem of the boundary contributions, since the geodesic curvature is zero. This calculation gives the same spectral problem as the vacuum energy calculation for the bosons and ghosts, but not the fermions and longitudinal modes.

5. Circular Wilson loop

Another case where the classical solution has an explicit simple form [47, 25] is a circular Wilson loop. Like the straight string case, this configuration is useful as it gives a laboratory to investigate many of the issues that arise in the case of the more general bent string configuration.

5.1 Classical solution and quadratic fluctuation action

The target space metric in polar coordinates is $(s = 2, 3)$[11]

$$ds^2 = \frac{1}{w^2}\left(dr^2 + r^2 d\phi^2 + dx^s dx^s + dw^2\right) + d\Omega_5^2\,. \tag{5.1}$$

We set $x^0 = \phi \in [0, 2\pi]$ and $x^1 = r \in [0, 1]$.

The classical solution and the induced metric are

$$w = \sqrt{1 - r^2}\,, \qquad g_{ij} = \begin{pmatrix} \frac{r^2}{w^2} & 0 \\ 0 & \frac{1}{w^4} \end{pmatrix}, \qquad \sqrt{g}g^{ij} = \begin{pmatrix} \frac{1}{rw} & 0 \\ 0 & rw \end{pmatrix}, \tag{5.2}$$

$$R^{(2)} = -2\,, \tag{5.3}$$

i.e. the world sheet metric is again that of AdS_2 [47].[12]

[11] Because of scale invariance, the radius of the circle may be set equal to one.

[12] To put the metric in a more standard form we set $y = w^{-1}$. Then $ds^2 = (y^2-1)^{-1}dy^2 + (y^2-1)d\phi^2$, or in terms of $\tanh\chi = r$, $ds^2 = d\chi^2 + \sinh^2\chi d\phi^2$.

In the conformal gauge the quadratic part of the bosonic action is (3.11), i.e.

$$S = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}D_i\zeta^a D_j\zeta^a + X_{ab}\zeta^a\zeta^b + g^{ij}D_i\zeta^q D_j\zeta^q\right), \tag{5.4}$$

where in the present case

$$X^{ab} = 2\delta^{ab} - g^{ij}\eta_i^a\eta_j^b\,, \qquad \eta_0 = \left(\frac{r}{w}, 0, 0, 0, 0\right), \qquad \eta_1 = \left(0, \frac{1}{w}, 0, 0, -\frac{r}{w^2}\right).$$

The nonzero components of the spin connection in the target space are

$$\Omega_0^{01} = 1\,, \qquad \Omega_0^{04} = -\frac{r}{w}\,, \qquad \Omega_1^{14} = \Omega_2^{24} = \Omega_3^{34} = -\frac{1}{w}\,, \tag{5.5}$$

so that all of the covariant derivatives are trivial, $D_i = \partial_i$, except for

$$D_0\zeta^0 = \partial_0\zeta^0 + \zeta^1 - \frac{r}{w}\zeta^4\,, \qquad D_0\zeta^1 = \partial_0\zeta^1 - \zeta^0\,, \qquad D_0\zeta^4 = \partial_0\zeta^4 + \frac{r}{w}\zeta^0\,,$$
$$D_1\zeta^1 = \partial_1\zeta^1 - \frac{1}{w}\zeta^4\,, \qquad D_1\zeta^4 = \partial_1\zeta^4 + \frac{1}{w}\zeta^1\,. \tag{5.6}$$

The covariant derivative ∇_i in the ghost action

$$S = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}\nabla_i\epsilon^\alpha\nabla_j\epsilon^\alpha - \frac{1}{2}R^{(2)}\epsilon^\alpha\epsilon^\alpha\right), \tag{5.7}$$

includes the world-sheet spin connection, whose only nonzero component is $\omega_0^{01} = w^{-1}$. This derivative is not the same as the covariant derivative in (5.6). However, if we rotate the fields

$$\begin{pmatrix}\tilde\zeta^1\\ \tilde\zeta^4\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\zeta^1\\ \zeta^4\end{pmatrix}, \qquad \cos\alpha = w\,, \qquad \sin\alpha = r\,, \qquad \frac{d\alpha}{dr} = \frac{1}{w}\,, \tag{5.8}$$

the mass matrix becomes diagonal, $\tilde X_{ab} = \mathrm{diag}\,(1,1,2,2,2)$, and the only nontrivial covariant derivatives are $D_0\zeta^0 = \partial_0\zeta^0 + w^{-1}\tilde\zeta^1$ and $D_0\tilde\zeta^1 = \partial_0\tilde\zeta^1 - w^{-1}\zeta^0$. Then the longitudinal modes ζ^0 and $\tilde\zeta^1$ again have the same action as the ghosts, leaving us with three massive and five massless transverse oscillations.

The same conclusion is reached by starting with the Nambu form of the action and choosing static gauge, where r and ϕ in (5.1) are identified with the world-sheet coordinates. Let us denote ξ^s, ξ^4, ξ^q the fluctuations of the x^s, w and the S^5 coordinates respectively. After rescaling $\bar\zeta^s = w^{-1}\xi^s$, the action is (here g_{ij} is the induced metric)

$$S = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}\partial_i\bar\zeta^s\partial_j\bar\zeta^s + 2\bar\zeta^s\bar\zeta^s + g^{ij}\partial_i\xi^4\partial_j\xi^4 + 2\xi^4\xi^4 + g^{ij}\partial_i\xi^q\partial_j\xi^q\right), \tag{5.9}$$

and the fields are normalized as

$$\|\xi\|^2 = \int dr\, d\phi\sqrt{g}\left(\bar\zeta^s\bar\zeta^s + \frac{1}{w^2}\xi^4\xi^4 + \xi^q\xi^q\right). \tag{5.10}$$

Note that the field ξ^4 (and ζ^4 above) is not normal to the surface, but $\tilde{\zeta}^4$ is. As explained in section 3.3, this choice of gauge has a non trivial ghost determinant (3.33). In the present case $h^{\parallel}_{ij} = \mathrm{diag}(1/w^2, r^2/w^2)$, so that

$$\Delta_{\mathrm{gh}} = \det{}^{1/2}(h^{\parallel}_{ik} h^{kj}) = \det{}^{1/2} w^2 \,. \tag{5.11}$$

The most natural way to regularize this determinant is by redefining the norm of the ξ^4, thus removing the extra normalization factor in (5.10). Then the result for the partition function in this gauge will be identical to the conformal gauge expression apart from the contributions of the ghosts and the longitudinal modes.[13]

Before gauge fixing, the fermionic lagrangian is (3.4)

$$L_{2\mathrm{F}} = -i\left(\sqrt{g} g^{ij}\delta^{IJ} - \epsilon^{ij} s^{IJ}\right) \bar{\theta}^I \rho_i D_j \theta^J \,. \tag{5.12}$$

To put this action in a 2-d covariant fermionic form in terms of the zweibein and spin connection of the induced metric we apply a local $\mathrm{SO}(1,9)$ rotation to transform the projected Dirac matrices ρ_i into constant Dirac matrices contracted with the induced zweibein as discussed in section 3.4. We get (cf. (3.37) and (3.6))

$$\begin{aligned} D_i\theta^J &= DD_i\theta^J - \frac{i}{2}\epsilon^{JK}\rho_i\theta^K \\ \rho_0 &= \eta_0^a \Gamma_a = \frac{r}{w}\Gamma_0 = e_0^\alpha S \Gamma_\alpha S^{-1} \\ \rho_1 &= \eta_1^a \Gamma_a = \frac{1}{w}\Gamma_1 - \frac{r}{w^2}\Gamma_4 = e_1^\alpha S\Gamma_\alpha S^{-1} \\ \mathcal{D}_0 &= \left(\partial_0 + \frac{1}{2}\Gamma_{01} - \frac{r}{2w}\Gamma_{04}\right) = S\,\hat{\nabla}_0 S^{-1} \\ \mathcal{D}_1 &= \left(\partial_1 - \frac{1}{2w}\Gamma_{14}\right) = S\,\hat{\nabla}_1 S^{-1} \,. \end{aligned} \tag{5.13}$$

The rotation matrix is

$$S = \exp\left(\frac{\alpha}{2}\Gamma_{14}\right), \tag{5.14}$$

with the same angle α as in (5.8). $\hat{\nabla}_i$ is the covariant derivative with spinor world-sheet connection,

$$\hat{\nabla}_0 = \partial_0 + \frac{1}{2w}\Gamma_{01}\,, \qquad \hat{\nabla}_1 = \partial_1 \,. \tag{5.15}$$

[13]If we were to expand the action without fixing the gauge $\zeta^0 = \zeta^1 = 0$, we would find the same action, but with ξ^4 replaced by $\tilde{\zeta}^4$, with the canonical normalization. The two longitudinal fluctuations ζ^0 and $\tilde{\zeta}^1$ drop out of the action. Then one could choose the gauge $\zeta^0 = \tilde{\zeta}^1 = 0$ (this is the gauge used in [7, 10] in the context of the bent string configuration). $\tilde{\zeta}^1$ and ζ^1 are related through a rotation by an angle $\cos\alpha = w$. This rotation introduces a jacobian which exactly cancels the ghost determinant in the former gauge. It is also easy to show directly that the ghost determinant is trivial in this gauge.

It is therefore natural to transform θ^I to the new variable Ψ^I

$$\theta^I = S\Psi^I \,. \tag{5.16}$$

Choosing the gauge $\Psi^1 = \Psi^2$, the fermionic lagrangian becomes

$$\begin{aligned} L_{2\mathrm{F}} &= -2i\sqrt{g}\left(g^{ij}\bar{\Psi}e_i^\alpha\Gamma_\alpha\partial_j\Psi + i\bar{\Psi}\Gamma_{01}\Psi\right) \\ &= -2i\bar{\Psi}\left(-\frac{1}{w^2}\Gamma_0\partial_0 + \frac{r}{w}\Gamma_1\partial_1 + i\frac{r}{w^3}\Gamma_{01}\right)\Psi \,. \end{aligned} \tag{5.17}$$

Here the 10-d Dirac matrices Γ_α play the role of world sheet Dirac matrices, as we can choose a representation in terms of the Pauli matrices $\Gamma_0 = i\sigma_2 \times I_8$, $\Gamma_1 = \sigma_1 \times I_8$. As in the case of the straight string (4.11), this is the action for a spinor of mass ± 1 in AdS_2.

Another natural way to fix the κ symmetry used in [48, 6][14]

$$\theta^1 = \Gamma_{0123}\theta^2 \,, \qquad \text{i.e.} \qquad \theta^1 = i\Gamma_4\theta^2 \,. \tag{5.18}$$

This gauge leads to the *same* result for the action as the $\theta^1 = \theta^2$ gauge as we shall explain below.[15] Expressing θ^2 in terms of $\theta^1 \equiv \theta$ one can check that

$$D_i\theta^1 = \frac{1}{\sqrt{w}}\left(\partial_i + \frac{1}{2}\Gamma_{i1}\right)\left(\sqrt{w}\,\theta\right), \tag{5.19}$$

and, in terms of

$$\vartheta \equiv \sqrt{w}\,\theta \,, \tag{5.20}$$

the lagrangian is

$$L_{2\mathrm{F}} = -2i\bar{\vartheta}\left[-\left(\frac{1}{w^3}\Gamma_0 - \frac{r}{w^3}\Gamma_4\right)\left(\partial_0 + \frac{1}{2}\Gamma_{01}\right) + \frac{r}{w}\Gamma_1\partial_1\right]\vartheta \,. \tag{5.21}$$

To simplify this expression, we again use a rotation, this time in the 0-4 plane. Define

$$\psi = \exp\left(\frac{\beta}{2}\Gamma_{04}\right)\vartheta \,, \tag{5.22}$$

where

$$\cosh\beta = \frac{1}{w} \,, \qquad \sinh\beta = \frac{r}{w} \,, \qquad \frac{d\beta}{dr} = \frac{1}{w^2} \,. \tag{5.23}$$

Then

$$L_{2\mathrm{F}} = -2i\bar{\psi}\left[-\frac{1}{w^2}\Gamma_0\left(\partial_0 + \frac{1}{2}\Gamma_{01}\right) + \frac{r}{w}\Gamma_1\partial_1 - \frac{r}{w^3}\Gamma_{104}\right]\psi \,. \tag{5.24}$$

---

[14] $\Gamma_{0123} = i\gamma_4 \times I_4 \times I_2$ in the notation of [5], where 10-d Dirac matrices are represented as $\Gamma^a = \gamma^a \times I_4 \times \sigma_1$, $a = 0, 1, 2, 3, 4$.

[15] Note that in the straight string case (4.1) this gauge is degenerate.

Though this action looks different from (5.17), it also describes a fermion of mass ± 1 (the mass term $\Gamma_{104} = i\Gamma_{23}$ commutes with Γ_0 and Γ_1, but is antihermitean, and its square is 1).

The normalization of ψ is

$$\|\psi\|^2 = \int dr d\phi \, \sqrt{g} \, w^{-1} \, \bar{\psi}\psi \, , \tag{5.25}$$

which is different from normalization of Ψ in (5.17). Like in the bosonic case, the difference of the normalizations of the fields in the two κ-symmetry gauges may be attributed to the difference in the corresponding ghost determinants. Indeed, if the gauge condition is $\theta^1 = H\theta^2$ where H is some matrix ($H = 1$ and $H = i\Gamma_4$ in the two gauges discussed above), then it is easy to find the ghost determinant corresponding to the transformation (3.18). In the cases we are interested in the case where the $\bar{x}$ background is constant on S^5 (i.e. when $\tilde{\rho}_i = \rho_i$)

$$\delta_\kappa \theta^1 = \rho_i^- k^i \, , \qquad \delta_\kappa \theta^2 = \rho_i^+ k^i \, , \tag{5.26}$$

where $\rho_i^\pm = (g_{ij} \pm e_{ij})\rho_j$, where $e^{ij} = \epsilon^{ij}/\sqrt{g}$ and k_i are unconstrained vector-spinor parameters, normalized as $\|k_i\|^2 = \int d^2\sigma \, \sqrt{g} \, g^{ij} \bar{k}_i k_j$. Then the ghost determinant is the inverse square root of the determinant of the spinor matrix[16]

$$g^{ij}(\rho_i^- - \rho_i^+ H^\dagger)(\rho_j^- - H\rho_j^+) \, . \tag{5.27}$$

Since $\rho_{(i}\rho_{j)} = g_{ij}$ this matrix is a trivial constant in the $\theta^1 = \theta^2$ gauge when $H = 1$, but, in general, it will depend on the components of the metric (i.e. on the coordinates) when $H \neq 1$. In the gauge $\theta^1 = i\Gamma_4\theta^2$ the resulting local ghost determinant should "compensate" (in an appropriate regularization scheme) for the difference in normalizations of the spinors Ψ in (5.17) and ψ in (5.24), explaining the equivalence of the results in the two κ-symmetry gauges.

5.2 Partition function

We thus end up with the same partition function (4.16) as in the straight string case, i.e. with that of a supersymmetric field theory on AdS_2. The result is again a constant whose precise value depends on regularization and measures for the fields.

This should not come as a surprise, since the circle and the straight line are related by a special conformal transformation, and the minimal surfaces also transform into each other. This is not to say that the partition functions should be identical,

[16] The ghost determinant can be obtained from the path integral

$$\Delta_{\rm gh} \int [dk_i] \exp \left[- \int d^2\sigma \sqrt{g} \, \bar{k}^i (\rho_i^- - \rho_i^+ H^\dagger)(\rho_j^- - H\rho_j^+) k_j \right] = 1 \, .$$

there is a subtle difference. Indeed, already the classical actions for a circle and a straight line are different. The reason can be traced to the inclusion of the point at infinity. The same subtlety should be present at the level of 1-loop partition function. In the case of the straight line it is natural to work with the strip model for AdS_2, while for the circle, the Poincaré disk is more natural. In calculating the determinant for the former we should include also functions that do not behave well at infinity, while in the circle case those should not be included. It is therefore probable that the calculation in appendix B is more appropriate for this problem rather than the straight string case.

6. 'Parallel lines'

Our main interest is in the minimal surface ending at the boundary of $AdS_5 \times S^5$ which is related to the correlation function of two anti-parallel Wilson loops. The minimal surface was constructed in [4] and accounts for the leading large-λ behavior ($c_0\sqrt{\lambda}/L$) of the "quark-anti-quark" (W-boson) potential in $\mathcal{N} = 4$ SYM theory. The first correction c_1/L to the potential will be given by the one-loop partition function of the type we study here. While some aspects of this computation were addressed before [6, 7, 10], our aim below will be to clarify some previously encountered problems and to set up a systematic framework which should allow to compute the finite numerical coefficient c_1.

6.1 The classical solution

In this section we will write the $AdS_5 \times S^5$ metric in terms of the coordinate $y = w^{-1}$ (cf. (3.3))

$$ds^2 = R^2 \left(y^2 dx^n dx^n + \frac{dy^2}{y^2} + d\Omega_5^2 \right) . \tag{6.1}$$

Here $n = 0, 1, 2, 3$ and we will use the index 4 to label the coordinate y. We will often set the radius R to be 1 in what follows.[17] If the Wilson lines are extended in the x^0 direction and located at $x^1 = \pm L/2$, the minimal surface is given by a function $y(x^1)$ (we use world-sheet coordinates $\sigma^i = x^i = (\tau, \sigma)$, $0 < \tau < T$).

The bosonic part of string action is then ($y' \equiv \partial_1 y$)

$$S = \frac{R^2}{2\pi\alpha'} T \int d\sigma \sqrt{y'^2 + y^4} . \tag{6.2}$$

The stationary point is determined by the second-order equation $yy'' = 4y'^2 + 2y^4$ with the first integral being

$$y'^2 = \frac{y^8}{y_0^4} - y^4 . \tag{6.3}$$

[17] To make the flat space limit explicit one should define the coordinate φ related to y by $y = R^{-1} e^{-\varphi/R}$. Then $ds^2 = e^{-2\varphi/R} dx^n dx^n + d\varphi^2 + R^2 d\Omega_5$ which becomes flat in the $R \to \infty$ limit.

y_0 is an integration constant, the minimal value of y. The special case of $y_0 = 0$ corresponds to the "straight string" configuration discussed in section 4. This special solution is a useful reference point: near the boundaries of the σ-interval it gives a good approximation to the general solution.

The induced metric is

$$g_{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^6 \end{pmatrix}, \qquad \sqrt{g} = y^4\,, \qquad \sqrt{g}g^{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^{-2} \end{pmatrix}. \tag{6.4}$$

The distance between the quark and anti-quark L is related to y_0 by [4]

$$L = \int_{-L/2}^{L/2} d\sigma = 2\int_{y_0}^{\infty} \frac{dy}{y^2\sqrt{y^4/y_0^4 - 1}} = \frac{\kappa_0}{y_0}\,, \tag{6.5}$$

$$\kappa_0 \equiv \frac{(2\pi)^{3/2}}{[\Gamma(1/4)]^2}\,. \tag{6.6}$$

We shall often set $y_0 = 1$ as the dependence on this parameter can be easily restored by rescalings ($\tau \to y_0^{-1}\tau$, i.e. $T \to T/y_0^2$). Then

$$y'^2 = y^8 - y^4\,, \qquad y'' = 4y^7 - 2y^3\,. \tag{6.7}$$

Let us first review how the classical contribution is computed. The action (6.2) takes the following value on the solution

$$S = \frac{R^2 T}{2\pi\alpha' y_0^2}\int_{-L/2}^{L/2} d\sigma\, y^4\,. \tag{6.8}$$

Since $(y^{-3}y')' = \frac{y^4+y_0^4}{y_0^2}$ is a total derivative (which goes to the boundary where $y = \infty$ and gives only a trivial divergence), we can replace y^4 by $-y_0^4$, assuming that the infinite boundary contribution should be dropped. This prescription is the same as normalizing the partition function to the straight line case (and is essentially equivalent to the one in [25]: the Legendre transform subtracts the same boundary term or total derivative[18]). Then

$$S = -\frac{R^2 T}{2\pi\alpha'}y_0^2\int_{-L/2}^{L/2} d\sigma = -\frac{R^2 T L}{2\pi\alpha'}y_0^2 = -\frac{\lambda^{1/2}(2\pi)^2}{[\Gamma(1/4)]^4}\frac{T}{L}\,, \tag{6.9}$$

where $\lambda = 4\pi g_s N = R^4/\alpha'^2$. This result is the same as in [4], here found in 'one shot' (without doing any further integrals).

[18]This is an example of an amusing relation. The Legendre transform can be written as an integral over a (rather complicated) total derivative. Instead, one can note that for smooth loops the Legendre transform which is equal to the divergence in the area is also equal (asymptotically) to the geodesic curvature K. Then we can use the Gauss-Bonnet theorem to write the action as $S = \frac{R^2}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\,(1 + \frac{1}{2}R^{(2)}) - \frac{R^2}{\alpha'}\chi$, where χ is the Euler number. Since $R^{(2)}$ approaches -2, the integral is manifestly convergent. For the present geometry $\chi = 0$.

In the flat space limit ($R \to \infty$) one finds that the quantum correction (2.10) to the potential vanishes because of the cancellation of the bosonic and fermionic contributions due to effective 2-d supersymmetry present after gauge fixing (see section 2.3). As was pointed out in [6], the 1-loop c_1/L correction to the effective potential may not, however, vanish in the present curved space case as there is no reason to expect that the action expanded near the solution $y = y(\sigma) \neq$ const. should have an effective world-sheet supersymmetry.

6.2 Quadratic fluctuations: bosons

In conformal gauge the bosonic action is (3.11), where $X_{pq} = 0$ and[19]

$$\begin{aligned} X^{ab} &= 2\delta^{ab} - g^{ij}\eta_i^a\eta_j^b\,, & \eta_0^a &= (y,0,0,0,0)\,, & \eta_1^a &= (0,y,0,0,y^{-1}y')\,, \\ D_i\zeta^a &= \partial_i\zeta^a + w_i^{ab}\zeta^b\,, & w_i^{ab} &= \partial_i x^\mu \Omega_\mu^{ab}\,, & w_i^{a4} &= y\delta_i^a = -w_i^{4a}\,. \end{aligned} \tag{6.10}$$

The ghost action is (3.26) or (5.7) where the covariant derivative ∇_i includes the world-sheet spin connection $\omega_0^{01} = y^{-3}y'$. The action contains the curvature $R^{(2)}$ of the induced metric g_{ij}

$$\sqrt{g}R^{(2)} = \left(\frac{1}{y^2}\right)'' , \qquad R^{(2)} = -2\left(1 + \frac{1}{y^4}\right). \tag{6.11}$$

Unlike the circle case, here there is no obvious rotation of the fields ζ^a such that the contribution of the longitudinal modes becomes the same as that of the ghosts.[20]

Choosing the static gauge in the Nambu action we denote by ξ^s ($s = 2,3$) the fluctuations of the two "longitudinal" 3-brane directions, by ξ^4 the fluctuation along the radial y-direction, and by ξ^q ($q = 5,\ldots,9$) the fluctuations in the 5-sphere directions. Their natural norms are

$$||\xi||^2 = \int d^2\sigma\sqrt{g}(y^2\xi^s\xi^s + y^{-2}\xi^4\xi^4 + \xi^q\xi^q)\,. \tag{6.12}$$

Introducing the rescaled fields[21]

$$\zeta^s = y\xi^s\,, \qquad \tilde{\zeta}^4 = y^{-3}\xi^4\,, \tag{6.13}$$

one finds (after integration by parts and use of the properties of the classical background (6.7)) the following expression for the quadratic fluctuation part of the gauge-

[19] In this form of the metric (6.1) the vielbein components are $E_n^m = y\delta_n^m$, $E_4^4 = y^{-1}$. If we restore the R dependence, $X_{ab} \to \frac{1}{R^2}X_{ab}$, then the "mass term" vanishes in the flat space limit as it should.

[20] In fact, the eigenvalues of the mass matrix X^{ab} are $(1,1,2,2,2)$, not $(-\frac{1}{2}R^{(2)}, -\frac{1}{2}R^{(2)}, 4 + R^{(2)}, 2, 2)$. In addition there are extra connection terms that remain after the rotation.

[21] It should be noted that redefinitions we make are accompanied by jacobians and thus do not introduce new quadratic or linear divergences.

fixed action[22]

$$S_{2\mathrm{B}} = \frac{1}{2}\int d^2\sigma\sqrt{g}\Big[g^{ij}\partial_i\zeta^s\partial_j\zeta^s + 2\eta^s\eta^s + g^{ij}\partial_i\tilde{\zeta}^4\partial_j\tilde{\zeta}^4 + (R^{(2)}+4)\tilde{\zeta}^4\tilde{\zeta}^4 + g^{ij}\partial_i\xi^q\partial_j\xi^q\Big]\,. \tag{6.14}$$

As follows from (6.12), the fields in (6.14) are normalized as follows

$$\|\xi\|^2 = \int d^2\sigma\sqrt{g}(\zeta^s\zeta^s + y^4\tilde{\zeta}^4\tilde{\zeta}^4 + \xi^q\xi^q)\,. \tag{6.15}$$

Thus, while the action $S_{2\mathrm{B}}$ seems to have a 'covariant' 2-d form with respect to the induced geometry (two massive scalars, one scalar with a potential, and 5 massless scalars in external 2-d metric), this is not true for the measure because of the y^4 factor in the $\tilde{\zeta}^4\tilde{\zeta}^4$ part. This is remedied by the inclusion of the ghost determinant (3.33), where in our case $h^{\parallel}_{ij} = \mathrm{diag}(y^2, y^2)$, so that

$$\Delta_{\mathrm{gh}} = \det{}^{1/2}\left(\frac{1}{y^4}\right)\,. \tag{6.16}$$

The most natural regularization of this determinant is achieved by rescaling the field $\tilde{\zeta}^4$, which cancels the renormalization factor in (6.15), much as in the case of the circle in section 5.1 (see (5.11)).

The same expression for the action (6.14) was found in [7]. There instead of $\tilde{\zeta}^4$ the authors used the fluctuating field normal to the surface $\eta^4 \equiv -\sin\alpha\zeta^1 + \cos\alpha\zeta^4 = -y^{-3}y'\xi^1 + y^{-3}\xi^4$, where α is the angle defined by $\cos\alpha = y^{-2}$, $\alpha' = 2y$, and $\zeta^a = E^a_\mu\xi^\mu$ are the target-space vielbein components of ξ^μ. The normalization of η^4 is canonical, and it is easy to see that the ghost determinant is trivial in that gauge.

6.3 Quadratic fluctuations: fermions

In the present case of the classical solution (6.3), which is constant in S^5 directions, one finds that the leading quadratic part of the fermionic action given by (3.4) depends on

$$\rho_0 = y\Gamma_0\,, \qquad \rho_1 = y\Gamma_1 + y^{-1}y'\Gamma_4\,, \qquad \mathcal{D}_i = \partial_i + \frac{1}{2}y\Gamma_i\Gamma_4\,, \tag{6.17}$$

where we used the fact that for AdS_5 space the non-vanishing components of the connection are $\Omega^{n4}_\mu = E^n_\mu$, $n = 0,1,2,3$. Then (3.4) becomes

$$L_{2\mathrm{F}} = i\left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}s^{IJ}\right)\left(\bar{\theta}^I\rho_i\mathcal{D}_j\theta^J - \frac{1}{2}i\epsilon^{JK}\bar{\theta}^I\rho_i\rho_j\theta^K\right)\,. \tag{6.18}$$

Here g_{ij} is the Minkowski version of the induced metric (6.4), i.e. the corresponding 2-d vielbein components e^α_i are

$$e^0_0 = y\,, \qquad e^1_1 = y^3\,, \qquad g_{ij} = \mathrm{diag}\left(-y^2, y^6\right)\,. \tag{6.19}$$

[22]As before, we absorb $1/2\pi\alpha'$ factor in the action into a redefinition of the fluctuation fields.

The crucial observation, that allows us to put the action (6.18) into a simple 2-d covariant form, is that the combination of Γ_1 and Γ_4 which appears in ρ_1 can be interpreted as a (local, σ-dependent) rotation of Γ_1 in the 1-4 plane

$$S\Gamma_1 S^{-1} = \cos\alpha\, \Gamma_1 + \sin\alpha\, \Gamma_4 = y^{-2}\Gamma_1 + y^{-4}y'\Gamma_4 = y^{-3}\rho_1\,, \tag{6.20}$$

where

$$S = \exp\left(-\frac{\alpha}{2}\Gamma_1\Gamma_4\right), \qquad \cos\alpha = y^{-2}\,, \qquad \sin\alpha = y^{-4}y'\,, \qquad \alpha' \equiv \frac{d\alpha}{d\sigma} = 2y\,. \tag{6.21}$$

Making the field redefinition

$$\theta^I \to \Psi^I \equiv S^{-1}\theta^I\,, \tag{6.22}$$

we then find that (6.18) takes the following simple '2-d covariant' form

$$L_{2\mathrm{F}} = i\left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}s^{IJ}\right)\left(\bar{\Psi}^I\tau_i\hat{\nabla}_j\Psi^J - \frac{1}{2}i\epsilon^{JK}\bar{\Psi}^I\tau_i\tau_j\Psi^K\right), \tag{6.23}$$

where τ_i play the role of the curved space 2-d Dirac matrices[23]

$$\tau_i = e_i^\alpha\Gamma_\alpha\,, \qquad \tau_0 = S^{-1}\rho_0 S = y\Gamma_0\,, \qquad \tau_1 = S^{-1}\rho_1 S = y^3\Gamma_1\,, \tag{6.24}$$

and $\hat{\nabla}_i$ is the 2-d curved space spinor covariant derivative corresponding to (6.19)

$$\hat{\nabla}_k = \partial_k + \frac{1}{4}\omega_k^{\alpha\beta}\Gamma_{\alpha\beta}\,, \qquad \hat{\nabla}_0 = \partial_0 + \frac{1}{2}y^{-3}y'\Gamma_0\Gamma_1\,, \qquad \hat{\nabla}_1 = \partial_1\,. \tag{6.25}$$

The lagrangian (6.23) is then

$$L_{2\mathrm{F}} = i\left(\sqrt{g}g^{ij} - \epsilon^{ij}\right)\bar{\Psi}^1\tau_i\hat{\nabla}_j\Psi^1 + i\left(\sqrt{g}g^{ij} + \epsilon^{ij}\right)\bar{\Psi}^2\tau_i\hat{\nabla}_j\Psi^2 - \epsilon^{ij}\bar{\Psi}^1\tau_i\tau_j\Psi^2\,. \tag{6.26}$$

It is easy to see that the covariant derivative and 'mass' terms here are separately invariant under the leading-order κ-symmetry transformations (see (3.18) and (5.26)) $\delta_\kappa\theta^I = \rho_i\kappa^{iI}$ or their 'rotated' form

$$\delta_\kappa\Psi^I = \tau_i\kappa'^{iI}\,, \qquad \kappa'^{iI} = S^{-1}k^{iI}\,. \tag{6.27}$$

Fixing the κ-symmetry gauge by the condition

$$\theta^1 = \theta^2\,, \qquad \text{i.e.} \qquad \Psi^1 = \Psi^2 \equiv \Psi\,, \tag{6.28}$$

[23] To prove eq.(6.23) one should note that $\tau_i = S^{-1}\rho_i S$ and that $\tilde{\mathcal{D}}_i \equiv S^{-1}\mathcal{D}_i S$ is found to be (see (6.21)) $\tilde{\mathcal{D}}_0 = \partial_0 + \frac{1}{2}y\Gamma_0(y^{-4}y'\Gamma_1 + y^{-2}\Gamma_4) = \hat{\nabla}_0 + B_0$, $\tilde{\mathcal{D}}_1 = \partial_1 - y\Gamma_1\Gamma_4 + \frac{1}{2}y\Gamma_1\Gamma_4 = \nabla_1 + B_1$, where $B_0 = \frac{1}{2}y^{-2}\tau_0\Gamma_4$, $B_1 = -\frac{1}{2}y^{-2}\tau_1\Gamma_4$. Finally, one observes that the connection B_i drops out from the action since $\tau^i B_i = 0$, $\epsilon^{ij}\tau_i B_j = 0$.

we get

$$L_{2F} = 2i\sqrt{g}\left(\bar{\Psi}\tau^i\hat{\nabla}_i\Psi + i\bar{\Psi}\tau_3\Psi\right), \tag{6.29}$$

$$\tau_3 \equiv \frac{\epsilon^{ij}}{2\sqrt{g}}\tau_i\tau_j = \Gamma_0\Gamma_1\,, \qquad (\tau_3)^2 = 1\,. \tag{6.30}$$

Note that choosing this gauge before the rotation, the action (6.18) may be written as

$$L_{2F} = 2i\sqrt{g}\bar{\theta}\rho^i\mathbf{D}_i\theta = i\sqrt{g}(\bar{\theta}\rho^i\partial_i\theta - \partial_i\bar{\theta}\rho^i\theta + ie_{ij}\bar{\theta}\rho^i\rho^j\theta)\,, \tag{6.31}$$

where $\mathbf{D}_j = \mathcal{D}_j + \frac{1}{2}ie_{jk}\rho^k$, $\rho^i = g^{ij}\rho_j$, $e_{jk} \equiv \frac{1}{\sqrt{g}}g_{jj'}g_{kk'}\epsilon^{j'k'}$. To see that the rotation (6.20), (6.21) is indeed a special case of the general rotation (3.37) discussed in section 3.4 we note that here the tangent t^μ_α and normal n^μ vector components are (here $\mu,\nu = 0,1,4$ label the target space AdS_3 coordinates inside AdS_5, i.e. $G_{\mu\nu} = (y^2, y^2, y^{-2})$):

$$t^\mu_{\hat{0}} = (1,0,0)\,, \qquad t^\mu_{\hat{1}} = (0, y^{-3}, y^{-3}y')\,, \qquad n^\mu = (0, -y^{-5}y', y^{-1})\,. \tag{6.32}$$

Then $\rho_\alpha = (\rho_{\hat{0}}, \rho_{\hat{1}})$ and $\rho_s \equiv \rho_\perp$ are (cf. (6.20))

$$\rho_{\hat{0}} = \Gamma_0\,, \qquad \rho_{\hat{1}} = y^{-3}\rho_1 = y^{-2}\Gamma_1 + y^{-4}y'\Gamma_4\,, \qquad \rho_\perp = y^{-3}\rho_1 = -y^{-4}y'\Gamma_4 + y^{-2}\Gamma_4\,, \tag{6.33}$$

so that

$$-S^{-1}dS = \frac{1}{4}(\rho_\alpha d\rho^\alpha + \rho_\perp d\rho_\perp) = \frac{1}{2}\Gamma_1\Gamma_4 d\alpha = y\Gamma_1\Gamma_4\,, \tag{6.34}$$

in agreement with (6.21). This is U(1) rotation that does not lead to a non-trivial jacobian in the present case of non-chiral 2-d fermions.

Another possible gauge choice is the analog of the covariantized light cone gauge of [15] (see (3.5) and (6.17)):

$$(\rho_0 + \rho_1)\Psi^1 = 0\,, \qquad (\rho_0 - \rho_1)\Psi^2 = 0\,, \tag{6.35}$$

or, explicitly, after the rotation (6.24),[24]

$$\tau_+\Psi^1 = (y\Gamma_0 + y^3\Gamma_1)\Psi^1 = 0\,, \qquad \tau_-\Psi^2 = (y\Gamma_0 - y^3\Gamma_1)\Psi^2 = 0\,. \tag{6.36}$$

The resulting action is essentially the same as (6.29) with left and right parts of Ψ explicitly separated.

Choosing a representation for Γ_a such that $\Gamma_{0,1}$ are 2-d Dirac matrices times a unit 8×8 matrix, i.e.[25]

$$\Gamma_0 = i\sigma_2 \times I_8\,, \qquad \Gamma_1 = \sigma_1 \times I_8\,, \qquad \tau_3 = \Gamma_0\Gamma_1 = \sigma_3 \times I_8\,, \tag{6.37}$$

[24] Note that this gauge choice is different from the one in [7] where instead of $\tau_\pm$ the combinations $\Gamma_0 \pm \Gamma_1'$ ($\Gamma_1' = S\Gamma_1 S^{-1}$) were used (the rotation was not explicitly done in [7]).

[25] Recall that the original 32×32 Dirac matrices are such that $\Gamma_a = \gamma_a \times I_4 \times \sigma_1$, or simply $\Gamma_a = \gamma_a \times I_4$ on a 16-subspace of left MW spinors, with $\gamma_{(a}\gamma_{b)} = \mathrm{diag}(-1,+1,+1,+1,+1)$. We are not distinguishing between Γ_a and γ_a, i.e. we treat Ψ as a 4-component spinor suppressing its extra 4 spectator indices.

where $\sigma_{1,2,3}$ are the Pauli matrices, we end up with 8 species of 2-d Majorana fermions living on a curved 2-d surface with a σ_3 mass term. Assuming that fermions are normalized with $\sqrt{g}$, the square of the resulting fermionic operator is then ($\tau_3^\dagger = \tau_3$)[26]

$$\Delta'_F = (D_F)^2 = (i\tau^i\hat{\nabla}_i - \tau_3)(i\tau^j\hat{\nabla}_j - \tau_3) = -\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1\,, \qquad (6.38)$$

where $\hat{\nabla}^2 = \frac{1}{\sqrt{g}}\hat{\nabla}^i(\sqrt{g}\hat{\nabla}_i)$.[27] Explicitly (recall that here we use the Minkowski signature)

$$\begin{aligned}\Delta'_F &= -\left[y^{-1}\left(\Gamma^0\partial_0 + \frac{1}{2}y^{-3}y'\Gamma_1\right) + y^{-3}\Gamma^1\partial_1\right]^2 + 1 \\ &= y^{-2}\left(\partial_0 + \frac{1}{2}y^{-3}y'\Gamma_0\Gamma_1\right)^2 - y^{-4}\partial_1\left(y^{-2}\partial_1\right) + \frac{1}{2}\left(1 - y^{-4}\right). \qquad (6.39)\end{aligned}$$

A similarly looking result for the fermionic operator was found in [7] where a different κ-symmetry gauge was used.[28]

One can also consider the quadratic fermionic action (6.18) in the "3-brane" gauge $\theta^1 = i\Gamma_4\theta^2$ (5.18). It was found in [6] that the sum of the quadratic fermionic terms in the action of [5] takes the simple form

$$\begin{aligned}S_{2\mathrm{F}} &= 2i\int d^2\sigma\left(\sqrt{g}g^{ij}y^2\bar{\vartheta}\Gamma_i\partial_j\vartheta - \epsilon^{ij}\partial_i y\bar{\vartheta}\Gamma^4\partial_j\vartheta\right) \\ &= 2i\int d^2\sigma\,(y^{-1/2}\bar{\theta})\left[\Gamma^1\partial_1 + (y^4\Gamma^0 + y'\Gamma^4)\partial_0\right](y^{-1/2}\theta)\,, \qquad (6.40)\end{aligned}$$

where $\theta \equiv \theta^1$ is the original GS target space spinor variable related to the rescaled field ϑ in [6] by $\theta = y^{1/2}\vartheta$. Here Γ^a are constant Dirac matrices ($\Gamma^0 = -\Gamma_0$, $\Gamma^m = \Gamma_m$, $m = 1, 4$). As was noted in [6], the resulting fermionic operator is non-degenerate.

At first sight, this operator is very different from the one in (6.29); in particular, it has no mass term. But the two are, in fact, closely related! To demonstrate this let us note that the combination of the Γ-matrices multiplying ∂_0 is actually a local Lorentz rotation of Γ^0 in the 0-4 plane with parameter β

$$\mathcal{S} = \exp\left(-\frac{\beta}{2}\Gamma^0\Gamma^4\right), \qquad \cosh\beta = y^2\,, \qquad \sinh\beta = y^{-2}y'\,, \qquad \beta' = 2y^3\,, \quad (6.41)$$

i.e.

$$\mathcal{S}\Gamma^0\mathcal{S}^{-1} = \cosh\beta\;\Gamma^0 + \sinh\beta\;\Gamma^4 = y^2\Gamma^0 + y^{-2}y'\Gamma^4\,. \qquad (6.42)$$

[26] Note that Dirac operator is self-adjoint with the measure $\sqrt{g}$.

[27] Since $\tau_i = e_i^\alpha\Gamma_\alpha$, where Γ_α are *constant*, and since $\hat{\nabla}$ is the covariant 2-d spinor derivative, the squaring relation is exactly the same as for the standard 2-d fermions in curved space.

[28] The kinetic part of our operator is actually different from the expression in [7] which contained an additional connection term in the covariant derivative, and our derivation of the action is much more straightforward. After submission of this paper S. Förste pointed out to us that the two expressions might be related to each other by a rotation, i.e. are equivalent.

Introducing

$$\chi = \mathcal{S}^{-1}\theta\,, \tag{6.43}$$

we get (note that $\bar\theta = \bar\chi\mathcal{S}^{-1}$ for any SO(9,1) rotation)

$$L_{2\mathrm{F}} = 2i\bar\chi\left[y\Gamma^0\partial_0 + y^{-1}\Gamma^1\partial_1 - \frac{1}{2}y^{-2}y'\Gamma_1 + y^2\Gamma^0\Gamma^1\Gamma^4\right]\chi\,. \tag{6.44}$$

To put this action into the 'curved space 2-d spinor' form we need to make a redefinition $\chi \to \psi$

$$\chi = y\psi\,, \tag{6.45}$$

$$L_{2\mathrm{F}} = 2i\bar\psi\left[y^3\Gamma^0\left(\partial_0 + \frac{1}{2}y^{-3}y'\Gamma_0\Gamma_1\right) + y\Gamma^1\partial_1 + y^4\Gamma^0\Gamma^1\Gamma^4\right]\psi\,. \tag{6.46}$$

Since $i\Gamma_4 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$, i.e.

$$\Gamma^0\Gamma^1\Gamma^4 = i\Gamma^2\Gamma^3\,, \tag{6.47}$$

we finally get, using (6.19) and (6.25), the expression that is essentially *equivalent* to (6.29)

$$L_{2\mathrm{F}} = 2i\sqrt{g}\left(\bar\psi\tau^i\hat\nabla_i\psi + i\bar\psi\tau_3'\psi\right), \qquad \tau_3' \equiv \Gamma_2\Gamma_3\,, \qquad (\tau_3')^2 = -1\,. \tag{6.48}$$

The square of the fermionic operator in (6.48) is indeed the same as in (6.38):

$$\Delta_F' = (D_F)^\dagger(D_F) = \left(i\tau^i\hat\nabla_i + \tau_3'\right)\left(i\tau^j\hat\nabla_j - \tau_3'\right) = -\hat\nabla^2 + \frac{1}{4}R^{(2)} + 1\,, \tag{6.49}$$

where we used that, in contrast to τ_3 in (6.29), τ_3' is *anti*hermitean and *commutes* with τ_i.

Since θ and thus its image under the rotation χ are assumed to have canonical normalization, $\|\chi\|^2 = \int d^2\sigma\sqrt{g}\bar\chi\chi$, we conclude that ψ in (6.45) should be normalized with the extra factor of y^2. As in the case of the circle in section 5.1, this extra factor is offset by the non-trivial κ-symmetry ghost determinant (5.27) corresponding to the gauge $\theta^1 = i\Gamma_4\theta^2$, so the fermionic contributions in the two κ-symmetry gauges are again equivalent.

6.4 Partition function

Let us first combine the bosonic contributions. In the conformal gauge

$$Z_{\text{bose, conf.g.}} = \frac{\det^{1/2}\left(-\nabla_{ij}^2 - \frac{1}{2}R^{(2)}g_{ij}\right)}{\det^{1/2}\left(-D_{ab}^2 + X_{ab}\right)\det^{5/2}\left(-\nabla^2\right)}\,, \tag{6.50}$$

while in the static gauge

$$Z_{\text{bose, stat.g.}} = \frac{1}{\det^{2/2}\left(-\nabla^2 + 2\right)\det^{1/2}\left(-\nabla^2 + R^{(2)} + 4\right)\det^{5/2}\left(-\nabla^2\right)}\,. \tag{6.51}$$

Up to global factors in the gauge group, the two expressions must be equivalent; it is easy to see, for example, that the corresponding logarithmic divergence coefficients are indeed the same

$$(b_2)_{\text{bose, conf.g.}} = (5+5-2) \times \frac{1}{6} R^{(2)} - 8 - R^{(2)}\,, \tag{6.52}$$

$$(b_2)_{\text{bose, stat.g.}} = (2+1+5) \times \frac{1}{6} R^{(2)} - 2 \times 2 - 4 - R^{(2)}\,, \tag{6.53}$$

where we used (6.10). The contributions of the massless determinants and the ghost determinant can be found, as usual, by integrating the conformal anomaly (see appendix A), but to find the massive determinants one needs to solve the corresponding spectral problems.

Including fermions, the expression for the 1-loop partition function of a string in $AdS_5 \times S^5$ background with world surface ending on two parallel lines is thus

$$Z^{\|}_{AdS_5 \times S^5} = \frac{\det^{8/2}(-\hat{\nabla}^2 + \frac{1}{4} R^{(2)} + 1)}{\det^{2/2}(-\nabla^2 + 2) \det^{1/2}(-\nabla^2 + R^{(2)} + 4) \det^{5/2}(-\nabla^2)}\,, \tag{6.54}$$

which is essentially the same as in [7]. Here we took the fermionic contribution in the $\theta^1 = \theta^2$ gauge, and the bosonic contribution in the static gauge.

The geometry under consideration is asymptotic to AdS_2. For example, if we change the coordinate σ to y the induced metric takes the form[29]

$$ds^2 = y^2 d\tau^2 + \frac{y^2}{y^4 - y_0^4} dy^2\,, \qquad y_0 \le y < \infty\,. \tag{6.55}$$

The $y_0 = 0$ limit of (6.55) corresponds to the straight string configuration where the metric becomes that of euclidean AdS_2 space (with $0 \le y < \infty$).

As usual in the case of negative curvature non-compact spaces similar to AdS, the corresponding actions will in general be divergent and one will need to add boundary counterterms. The details of how the divergent boundary behavior is properly accounted for is actually rather irrelevant for the purpose of extracting the non-trivial finite T/L part of $\ln Z^{\|}_{AdS_5 \times S^5}$ we are interested in.

To avoid altogether questions about boundary terms (and details of topological infinity cancellation) we may normalize our partition function for each field by the partition function of an equivalent field in the straight string configuration, i.e. divide the partition function (6.54) for the noncompact hyperbolic negative curvature space (6.55) by the partition function (twice, to account for the two asymptotic regions), (4.16) for the AdS_2 case

$$\bar{Z}^{\|}_{AdS_5 \times S^5}(y_0) = \frac{Z^{\|}_{AdS_5 \times S^5}(y_0)}{Z^{\|}_{AdS_5 \times S^5}(0)}\,. \tag{6.56}$$

[29] It is sometimes useful to use the coordinate $w = 1/y$, in terms of which the metric is $ds^2 = \frac{1}{w^2}[d\tau^2 + \frac{w_0^4}{w_0^4 - w^4} dw^2]$, where $0 < w < w_0 = 1/y_0$ and $w = 0$ corresponds to the boundary.

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Since the topology and the near-boundary (large y) behavior of the two metrics is the same, this eliminates the problem of carefully tracking down all boundary terms in the expressions for the determinants and allows us to ignore the boundary contributions as well as the total derivative bulk terms (such as the logarithmically divergent terms proportional to $\int d^2\sigma\sqrt{g}R^{(2)}$).[30]

The ratio of the determinants for the metric (6.55) and for its $y_0 = 0$ limit will be finite and well-defined. This is actually the standard recipe of defining the determinants of Laplace operators on (e.g. 2-dimensional) non-compact spaces by using fiducial metrics of constant negative curvature which have the same topology and asymptotic behavior. As a result, one will need to compute only the well-defined heat kernels like $\mathrm{Tr}[e^{-t\Delta(y_0)} - e^{-t\Delta(0)}]$.

From a practical point of view, the subtraction of the AdS_2 contributions allows us, as in the evaluation of the classical action (6.8), (6.9), to freely integrate by parts and to drop all divergent boundary contributions. Some examples illustrating this procedure are given in appendix A.3.

6.5 Crude approximation for the 1-loop potential

One may make the following very simple (but probably too crude) estimate of the value of the resulting partition function and thus of the coefficient in the 1-loop correction c_1/L to the potential. The classical solution y as a function of σ is approximately equal to y_0 and changes slowly near $\sigma \approx 0$ and then blows up to infinity at the boundaries of the σ-interval $(-L/2, L/2)$. It seems reasonable to assume that the near-boundary behavior of $y(\sigma)$ should not be very important for the value of the normalized partition function (6.56). One may then approximate $y(\sigma)$ to be made of three straight pieces. A part where $y \approx y_0$, $y' \approx 0$, $y'' \approx 0$, and two parts connecting it to the boundary. Note that this is *not* the same as taking the flat space limit, since now in (6.54) we have determinants of operators with non-zero mass.

Below we will estimate the contribution of the part at $y \approx y_0$. We did not evaluate the contribution from the two pieces connecting it to the boundary.

Since y is assumed to change very slowly, we may set $R^{(2)} \approx 0$. Then we are left with the following combination of determinants in flat metric[31]

$$W = \frac{1}{2}\Big[2\ln\det(-\partial^2 + 2y_0^2) + \ln\det(-\partial^2 + 4y_0^2) + 5\ln\det(-\partial^2) - 8\ln\det(-\partial^2 + y_0^2)\Big]. \tag{6.57}$$

This effective action is UV finite because of the obvious mass sum rule. The non-zero finite result for W can probably be interpreted as the vacuum energy of some spontaneously broken supersymmetric 2-d field theory corresponding to (6.57).

[30] It is easy to see that the divergent integral $\int d^2\sigma\sqrt{g}R^{(2)}$ gets contribution only from the boundary behavior of the metric.

[31] Contributions of overall constant factors like y_0^{-2} in the operators cancel out due to supersymmetric balance of the numbers of fields.

Assuming Dirichlet boundary conditions in both τ and σ directions and taking $T \to \infty$ (so that we can integrate over the continuous eigenvalue in τ-direction) we get[32]

$$\frac{1}{2}\ln\det(-\partial^2 + m^2) - \frac{1}{2}\ln\det(-\partial^2) = \frac{1}{2}T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\ln\left(1+\frac{\omega_n^2}{k^2}\right)$$
$$= \frac{1}{2}T\sum_{n=0}^{\infty}\omega_n\,, \qquad \omega_n^2 = \left(\frac{\pi n}{L}\right)^2 + m^2\,. \quad (6.58)$$

Thus

$$W = z_1\frac{\pi}{2}\frac{T}{L}\,, \qquad z_1 = \sum_{n=1}^{\infty}\left[\sqrt{n^2+4a^2} + 2\sqrt{n^2+2a^2} + 5\sqrt{n^2} - 8\sqrt{n^2+a^2}\right], \quad (6.59)$$

$$a = \frac{Ly_0}{\pi} = \frac{2\sqrt{2\pi}}{[\Gamma(1/4)]^2} \approx 0.38138\,, \quad (6.60)$$

where we have used (6.5). To compare, in the "massless" case one finds [23] (see (2.10)) $z_1 = \sum_{n=1}^{\infty} n = \zeta(-1) = -1/12 \approx -0.08333$. The infinite sum (6.59) is *convergent* and its the numerical evaluation gives

$$z_1 \approx -1.24966\,. \quad (6.61)$$

Thus the coefficient of the $1/L$ potential is negative, i.e. has the same sign as a boson in flat space.

To this one has to add the contribution of the two "flat" lines connecting it to the boundary. This will give a result that is not identical to that of AdS_2, because these lines extend only up to y_0.

To go beyond the above crude approximation and to compute $\ln \bar{Z}^{||}_{AdS_5\times S^5} = -c_1\frac{T}{L}$ in (6.56) exactly one may use the following strategy: (i) first, one may compute the contributions of the massless determinants and the jacobian using the results of appendices A and C; (ii) then, since the induced 2-d metric is conformally flat, one may rescale it to the flat space one, isolating the conformal anomaly parts of the determinants; (iii) finally, one may compute the spectra of the resulting operators in flat metric with y-dependence being only in the mass terms. For example, the bosonic operators in (6.54) then become

$$-\partial_0^2 - \partial_1'^2 + 2y^2\,, \qquad -\partial_0^2 - \partial_1'^2 + 2y^2 - 2y^{-2}\,, \qquad -\partial_0^2 - \partial_1'^2\,, \quad (6.62)$$

where we have made the coordinate change $\sigma \to \sigma'$ such that the 2-d metric becomes conformally flat,

$$ds^2 = y^2(d\tau^2 + d\sigma'^2)\,, \qquad d\sigma' = y^2 d\sigma\,. \quad (6.63)$$

The computation of the spectra of these operators defined in the 2-d strip (T, L'), where $L' = [\Gamma(1/4)]^4 L/8\pi^2$ is the range of σ' (see appendix A.3), is left for the future.

[32] In general, for a massive determinant we get $\prod_{n,k=1}^{\infty}\left[(\frac{\pi n}{T})^2 + (\frac{\pi k}{L})^2 + m^2\right]$. For a massless determinant $\prod_{n,m=1}^{\infty}\left[(\frac{\pi n}{T})^2 + (\frac{\pi m}{L})^2\right] = (2L)^{-1/2}\eta(i\frac{T}{L})$, where $\eta(x) = e^{i\frac{\pi}{12}x}\prod_{n=1}^{\infty}(1 - e^{i\pi n x})$.

7. Conclusions

We have presented a systematic treatment of the Green Schwarz string in curved $AdS_5 \times S^5$ space. We found the quadratic fluctuation operators in conformal and static gauges (for Polyakov and Nambu-Goto actions). A careful treatment was presented of the measure factors and ghost determinants.

We also considered two different ways of gauge fixing the κ-symmetry, and explained how one can relate the GS fermion kinetic term to the standard 2-d Dirac fermion action on a curved 2-d background by making a local target-space Lorentz rotation.

We discussed the resolution of the problem of the logarithmic $O(R^{(2)})$ divergences encountered (in the Nambu framework) in [7]. First, as in the case of the flat target space, the divergence proportional to $\int R^{(2)}$ should be accompanied by a boundary term, promoting it to the Euler number, and thus is topological. The cancellation of topological divergences in a critical string theory is implied by careful definition of the path integral measure. This is clearer in the Polyakov approach, but should also be true in the Nambu formulation. In any case, this issue does not arise in the case of the induced 2-d geometries that we discussed (except for the circle), since there the Euler number vanishes.

We have emphasized that the natural way to define the partition function in the case where the induced geometry is asymptotic to AdS_2 (the case relevant for computing the correction to quark-anti-quark potential) is to normalize with respect to the partition function for AdS_2 space. Then the issues of boundary counterterms and divergences simply do not arise.

We have studied the cases of minimal surfaces ending along a straight line, a circle and two lines. In the first two cases we ended up with a supersymmetric field theory on AdS_2.

The straight line is a BPS object, and therefore one expects $Z = 1$, as we were able to verify. We presented two other ways of calculating the partition function on AdS_2, which give different results. The discrepancy is attributed to different regularizations and to assumptions about the asymptotics of the eigenfunctions. Those calculations might be more appropriate for the circular loop geometry, where supersymmetry is broken.

In the case of the two parallel lines we have found the general expression for the partition function and showed how to express it in terms of the determinants of 2-d Laplace operators on a flat strip with potentials depending only on one of the two coordinates. We have not, however, addressed the issue of finding exact analytical or numerical methods of computing the corresponding determinants and thus the value of the numerical constant in the subleading correction to the $1/L$ potential.

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A. The dependence of determinants on measure and conformal factors

A.1 Bosonic operators

We start with a review of some general facts about divergences and conformal anomalies of 2-d determinants. In particular, we review the issue of the measure dependence of the determinants [49, 50]. The action

$$S_2 = \frac{1}{2}\int d^2\sigma\sqrt{g}(g^{ij}\partial_i\phi\partial_j\phi + X\phi^2) \equiv (\phi, \Delta\phi)\,,$$
$$(\phi, \phi') \equiv \int d^2\sigma\sqrt{g}M(\sigma)\phi(\sigma)\phi'(\sigma)\,, \tag{A.1}$$

where the scalar product is defined with an extra measure factor M, implies that the relevant Laplace operator that occurs in the determinant is, not Δ, but rather

$$\Delta_M = M^{-1}(-\nabla^2 + X)\,. \tag{A.2}$$

The dependence of $\det\Delta_M$ on M can be determined [49] by using the standard observation that, since $\delta\Delta_M = -(M^{-1}\delta M)\Delta_M$, the variation of

$$\ln\det\Delta_M = -\int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\,\mathrm{Tr}\exp(-t\Delta_M)\,, \tag{A.3}$$

with respect to $\ln M$ can be expressed in terms of the Seeley coefficients of Δ_M. It is then easy to see that only the quadratically (and linearly) divergent and finite parts of $\ln\det\Delta_M$ are dependent on M, but that the logarithmically divergent part is M-independent. This is in agreement with naive expectation that the M dependence should be given by $\ln M$ multiplying a regularized "$\delta(0)$".

Equivalently, note that (A.2) can be written as

$$\Delta_M = -\tilde{\nabla}^2 + \tilde{X}\,, \qquad \tilde{g}_{ab} \equiv Mg_{ab}\,, \qquad \tilde{X} \equiv M^{-1}X\,. \tag{A.4}$$

Then the divergent part of this determinant is given by the standard expression [51]

$$
\begin{aligned}
(\ln\det\Delta_M)_\infty = &-\frac{1}{4\pi}\Lambda^2\int d^2\sigma\sqrt{\tilde g}\pm\frac{1}{4\sqrt{\pi}}\Lambda\int ds\sqrt{\tilde\gamma}-\\
&-\frac{1}{4\pi}\ln\Lambda^2\left[\int d^2\sigma\sqrt{\tilde g}\left(\frac{1}{6}\tilde R-\tilde X\right)+\frac{1}{3}\int ds\sqrt{\tilde\gamma}\tilde K\right], \qquad \text{(A.5)}
\end{aligned}
$$

where $\pm$ corresponds to the Dirichlet and Neumann boundary conditions, $\tilde\gamma = M\gamma$ is the boundary metric and K is the trace of the second fundamental form. It is easy to see that the dependence on M drops out of the $\ln\Lambda$ term.[33]

The dependence of the finite part of $\ln\det\Delta_M$ on M is dictated by the same Seeley coefficient a_2 which multiples the $\ln\Lambda$ term and determines the conformal anomaly. This coefficient, a_2, appears in the $t\to 0$ expansion of the heat kernel,

$$
\mathrm{Tr}[F(\sigma)\exp(-t\Delta_M)] = \frac{1}{t}a_0(F|\Delta_M)+\frac{1}{\sqrt{t}}a_1(F|\Delta_M)+a_2(F|\Delta_M)+O\left(\sqrt{t}\right), \qquad \text{(A.6)}
$$

where F is an arbitrary function, and

$$
\begin{aligned}
a_0(F|\Delta_M) &= \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}M(\sigma)F(\sigma)\\
a_1(F|\Delta_M) &= \mp\frac{1}{8\sqrt{\pi}}\int ds\,\sqrt{\gamma}\sqrt{M(s)}F(s) \qquad \text{(A.7)}\\
a_2(F|\Delta_M) &= \frac{1}{4\pi}\left[\int d^2\sigma\,\sqrt{g}F(\sigma)b_2(\Delta_M)+\int ds\,\sqrt{\gamma}\left(F(s)c_2(\Delta_M)\mp\frac{1}{2}\sqrt{\gamma}\partial_n F\right)\right].
\end{aligned}
$$

Here

$$
\begin{aligned}
b_2(\Delta_M) &= \frac{1}{6}R^{(2)}-X-\frac{1}{6}\nabla^2\ln M\\
c_2(\Delta_M) &= \frac{1}{3}\left(K-\frac{1}{2}\partial_n\ln M\right). \qquad \text{(A.8)}
\end{aligned}
$$

and ∂_n is the inward pointing derivative normal to the boundary. The signs $\mp$ are for D and N boundary conditions. The dependence of the finite part on the measure factor M is found by integrating the equation (we assume that Δ_M has a trivial kernel)

$$
\delta(\ln\det\Delta_M) = -a_2(\delta\ln M|\Delta_M), \qquad \text{(A.9)}
$$

i.e.

$$
\begin{aligned}
(\ln\det\Delta_M)_{\rm fin} = &(\ln\det\Delta_1)_{\rm fin}-\\
&-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left[\ln M\left(\frac{1}{6}R^{(2)}-X\right)+\frac{1}{12}\partial^i\ln M\partial_i\ln M\right]-\\
&-\frac{1}{12\pi}\int ds\sqrt{\gamma}\left(\ln MK\mp\frac{1}{2}\partial_n\ln M\right). \qquad \text{(A.10)}
\end{aligned}
$$

[33]This is not unexpected since the relevant part of the logarithmic divergence is proportional to the Euler number of the $\tilde g$ metric but it should be the same as the Euler number of the g metric. Strictly speaking, this is so if M is smooth inside the domain, so as not to change the topology.

A.2 Fermionic operators

Consider now the fermionic action

$$\int d^2\sigma\sqrt{g}\,\bar{\psi}e^i_\alpha\tau^\alpha D_i\psi = \int d^2\sigma\sqrt{g}\,\bar{\psi}D_F\psi\,, \tag{A.11}$$

where τ_α are 2-d Dirac matrices. Assume that the norm contains an extra function $\mathcal{K}$

$$\|\psi\|^2 = \int d^2\sigma\sqrt{g}\,\mathcal{K}\ \bar{\psi}\psi\,. \tag{A.12}$$

Then the relevant second order operator is

$$\Delta^{(F)}_{\mathcal{K}} = (\mathcal{K}^{-1}D_F)^2 = \mathcal{K}^{-2}(-\hat{\nabla}^2+\cdots)\,. \tag{A.13}$$

While $\Delta^{(F)}_{\mathcal{K}}$ looks like the Laplace operator (A.2), there are two important differences compared to the scalar case: (i) the fermionic measure is $\mathcal{K}$, not $M=\mathcal{K}^2$, and (ii) in addition to the overall factor $\mathcal{K}^{-2}$, there is also an extra first derivative term, leading to an extra connection and extra potential terms.

The dependence on $\mathcal{K}$ can be found using again the variational argument, as in the derivation of conformal anomaly. The logarithmic divergences again do not depend on $\mathcal{K}$. The variation of $(\mathcal{K}^{-1}D_F)^2$ over $\mathcal{K}$ is the same as of $\mathcal{K}^{-2}(D_F)^2$, but now it is the Seeley coefficient of $(\mathcal{K}^{-1}D_F)^2$ that is to be used in (A.9).

In more detail, set $\mathcal{K}=e^\lambda$ and choose the conformal frame $e^\alpha_i = e^\rho\delta^\alpha_i$. Then it is easy to show that the since the spinor derivative is $D_j=\partial_j+\frac{1}{2}i\tau_3\epsilon_{jk}\partial_k\rho$, where the index contractions are with respect to the flat metric, the operator $(\mathcal{K}^{-1}D_F)^2$ becomes (we add here a potential term Y for generality)

$$\begin{aligned}\Delta^{(F)}_{\mathcal{K}} &= -\left(e^{-\lambda}e^{-\frac{3}{2}\rho}\tau_i\partial_i e^{\frac{1}{2}\rho}\right)^2 + e^{-2\lambda}Y \\ &= -e^{-2\lambda-2\rho}\left(\delta^{ij}+i\epsilon^{ij}\tau_3\right)\partial_i\left(1-\lambda-\frac{1}{2}\rho\right)\partial_j\left(1+\frac{1}{2}\rho\right)+e^{-2\lambda}Y\,,\end{aligned} \tag{A.14}$$

where ∂_i and ∂_j act on all terms to the right. This can be put into the standard form (A.2) (with $M=\mathcal{K}^2$) as follows

$$\begin{aligned}\Delta^{(F)}_{\mathcal{K}} &= e^{-2\lambda}\left[-e^{-2\rho}(\partial_k+B_k)^2+X\right] \\ \partial_k+B_k &= \partial_k+\frac{i}{2}\epsilon_{kj}\tau_3(\partial_j\rho+\partial_j\lambda)-\frac{1}{2}\partial_k\lambda = D_k-\frac{1}{2}\tau_j\tau_k\partial_j\lambda \\ X &= Y-\frac{1}{2}e^{-2\rho}\partial^2\rho-\frac{1}{2}e^{-2\rho}\partial^2\lambda = Y+\frac{1}{4}R^{(2)}-\frac{1}{2}\nabla^2\lambda\,,\end{aligned} \tag{A.15}$$

where we have used that in $d=2$ $\tau^i\tau_j\tau_i=0$.

The corresponding Seeley coefficient is thus (cf. (A.8))

$$b_2\left(\Delta^{(F)}_{\mathcal{K}}\right) = \left(\frac{1}{6}-\frac{1}{4}\right)R^{(2)}-Y-\left(\frac{1}{3}-\frac{1}{2}\right)\nabla^2\lambda\,, \tag{A.16}$$

or, in conformal coordinates,

$$\sqrt{g}\, b_2\left(\Delta_{\mathcal{K}}^{(F)}\right) = -\left(\frac{1}{3}-\frac{1}{2}\right)\partial^2\rho - \left(\frac{1}{3}-\frac{1}{2}\right)\partial^2\lambda - \sqrt{g}Y\,, \tag{A.17}$$

so that λ enters like the conformal factor (this argument determines the conformal anomaly of the Dirac operator since $\sqrt{g}\, e^i_\alpha = e^\rho \delta^i_\alpha$). As in (A.9), we have (omitting obvious boundary terms)

$$\delta\left(\ln\det \Delta_{\mathcal{K}}^{(F)}\right) = -2a_2\left(\delta \ln \mathcal{K} | \Delta_{\mathcal{K}}^{(F)}\right), \tag{A.18}$$

and thus

$$\begin{aligned}\left(\ln\det \Delta_{\mathcal{K}}^{(F)}\right)_{\text{fin}} &= \left(\ln\det \Delta_{1}^{(F)}\right)_{\text{fin}} + \\ &\quad + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left[\ln \mathcal{K}\left(\frac{1}{6}R^{(2)} + 2Y\right) + \frac{1}{6}g^{ij}\partial_i \ln\mathcal{K}\partial_j\ln\mathcal{K}\right].\end{aligned} \tag{A.19}$$

Note that the conformal anomaly of a scalar is twice as of a 2-d fermion, since for a scalar $M \to e^{2\rho}$ while for a fermion $\mathcal{K} \to e^{\rho}$. The anomaly of a GS spinor is 4 times as much as of 2-d fermion since here one needs to take $\mathcal{K} \to e^{2\rho}$ on top of the flat space operator.[34]

It is useful to compare this with the result found by treating $\Delta_{\mathcal{K}}^{(F)}$ as a scalar operator (A.8) with $M = \mathcal{K}^2$. According to (A.10) we would get (using that $X \to \frac{1}{4}R^{(2)} + X$ in this case)

$$\begin{aligned}\left(\ln\det \Delta_{\mathcal{K}^2}\right)_{\text{fin}} &= \left(\ln\det \Delta_{1}\right)_{\text{fin}} - \\ &\quad - \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left[\ln \mathcal{K}\left(-\frac{1}{6}R^{(2)} - 2X\right) + \frac{1}{3}g^{ij}\partial_i \ln\mathcal{K}\partial_j\ln\mathcal{K}\right].\end{aligned} \tag{A.20}$$

Thus

$$\left(\ln\det \Delta_{\mathcal{K}^2}\right)_{\text{fin}} - \left(\ln\det \Delta_{\mathcal{K}}^{(F)}\right)_{\text{fin}} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left[\left(\frac{1}{3}+\frac{1}{6}\right)g^{ij}\partial_i \ln\mathcal{K}\partial_j\ln\mathcal{K}\right]. \tag{A.21}$$

As expected, there is a non-trivial difference for a non-constant $\mathcal{K}$.

A.3 Explicit results for some determinants

As was mentioned in the text, one way to calculate the determinants in a curved geometry is to transform to flat metric. Instead of a complicated kinetic term depending on induced metric one then has to deal with a complicated mass term.

[34] Note that redefining spinors with careful account of measure factors gives equivalent results. For example, the usual 2-d spinor action is, in conformal gauge, $\int d^2\sigma\sqrt{g}\bar\psi e^{-\frac{3}{2}\rho}\tau^\alpha\partial_\alpha e^{\frac{1}{2}\rho}\psi$ with the measure $\int d^2\sigma\sqrt{g}\bar\psi\psi$. Redefining $\psi' = e^{\frac{1}{2}\rho}\psi$ we get $\int d^2\sigma\sqrt{g}\bar\psi' e^{-2\rho}\tau^\alpha\partial_\alpha\psi'$ with the measure $\int d^2\sigma\sqrt{g}e^{-\rho}\bar\psi'\psi'$, i.e. $\mathcal{K} = e^{-\rho}$. That corresponds to the operator $\mathcal{K}^{-1}D_F = e^{-\rho}\tau^\alpha\partial_\alpha$ which has the same determinant as $e^{-\frac{3}{2}\rho}\tau^\alpha\partial_\alpha e^{\frac{1}{2}\rho}$.

Let us consider the expression for a scalar in the general "bent" string configuration of section 6. The determinant consists of two terms: a flat-space determinant and a conformal anomaly part. For a single massless scalar with canonical normalization the conformal anomaly related part of its bulk effective action is

$$\begin{aligned} \mathcal{W} &= \frac{1}{2}\ln\det(-\nabla^2) - \frac{1}{2}\ln\det(-\partial^2) \\ &= -\frac{1}{24\pi}\left[\ln\Lambda\int d^2\sigma\sqrt{g}R^{(2)} + \frac{1}{4}\int R^{(2)}(-\nabla^{-2})R^{(2)}\right] \\ &\to -\frac{1}{24\pi}\int d^2\sigma'\partial_\alpha\rho\partial_\alpha\rho\,, \end{aligned} \tag{A.22}$$

where we have chosen the conformal coordinate system where

$$ds^2 = e^{2\rho}(d\tau^2 + d\sigma'^2)\,, \qquad \rho = \ln y\,, \qquad d\sigma' = \frac{y^2}{y_0^2}d\sigma\,. \tag{A.23}$$

To evaluate this integral let us note that at the boundary $y = \infty$. As a result, the total derivative and boundary contributions are trivial divergences which can be 'renormalized away' by subtracting the expression for the straight line case ($y_0 \to 0$). This allows us to freely integrate by parts and to replace, e.g., $\int y^4 \to -\int y_0^4$. Dropping the total derivative part we thus get

$$\mathcal{W} = -\frac{1}{24\pi}\int d^2\sigma'\partial_i\rho\partial_i\rho = -\frac{1}{24\pi}\int d\tau d\sigma\, y_0^2\frac{y'^2}{y^4} \qquad \to \qquad \frac{TL}{12\pi}y_0^2 = \frac{2\pi^2}{3[\Gamma(1/4)]^4}\frac{T}{L}\,. \tag{A.24}$$

Eq. (A.24) is to be added to the flat space result (2.10) (we assume Dirichlet boundary condition)

$$\frac{1}{2}\ln\det(-\partial^2) = -\frac{\pi}{24}\frac{T}{L'}\,. \tag{A.25}$$

Here L' is the range of σ' which is different from L by a factor,

$$L' = \int d\sigma\,\frac{y^2}{y_0^2} = \frac{2\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)}\frac{1}{y_0} = \frac{[\Gamma(1/4)]^2}{2\sqrt{2\pi}}\frac{1}{y_0} = \frac{[\Gamma(1/4)]^4}{2(2\pi)^2}L\,, \tag{A.26}$$

where we used that $\Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi$. Finally, we get the following expression for the massless scalar determinant

$$\frac{1}{2}\ln\det(-\nabla^2) = -\frac{(\pi-2)\pi^2}{3[\Gamma(1/4)]^4}\frac{T}{L}\,. \tag{A.27}$$

Like the flat-space potential (2.10) and like the tree-level potential (6.9) this expression is *negative*.

Multiplied by 5, this gives the result for the contribution of massless fluctuations in the S^5 directions to the partition function (6.54). Other determinants should lead to similar contributions.

Let us now consider the case of the fermionic determinant using (A.19). One finds that the conformal anomaly for a massless 2-d spinor is 1/2 of that for the scalar (A.24), i.e.

$$\mathcal{W}_{\rm F} = -\frac{1}{48\pi}\int d^2\sigma' \partial_\alpha\rho\partial_\alpha\rho = \frac{\pi^2}{3[\Gamma(1/4)]^4}\frac{T}{L}\,. \tag{A.28}$$

B. Partition function in straight string case (AdS_2)

Here we describe a direct approach to the calculation of the partition function (4.16) on AdS_2 which complements the discussion in section 4.3.

B.1 Spectral density and ζ-function on Poincaré disc

Our starting point will be the spectrum of the operator (4.17) with Dirichlet conditions at the boundary of the Poincaré disc, following [52].

The trace of heat kernel is defined by

$$K_B(t;m^2) = \mathrm{Tr}\exp[-t(-\Delta+m^2)] = \frac{1}{2\pi^2}\int_0^\infty \exp\left[-t\lambda(\nu)\right]\mu(\nu)d\nu\,, \tag{B.1}$$

and the zeta function is

$$\zeta_B(s;m^2) = \mathrm{Tr}\,\frac{1}{(-\Delta+m^2)^s} = \frac{1}{2\pi^2}\int_0^\infty \frac{\mu(\nu)d\nu}{\lambda(\nu)^s}\,. \tag{B.2}$$

The density of states for a scalar laplacian $-\Delta+m^2$ (in our case $m^2=0,2$) is

$$\mu_B(\nu) = \pi\nu\tanh(\pi\nu)\,, \tag{B.3}$$

where the eigenvalues of the laplacian are $\lambda = \nu^2+m^2+1/4$. If we plug in this density of states (B.3) into (B.2) we find (dropping the divergence)

$$\zeta_B(s;m^2) = \frac{(m^2+1/4)^{(1-s)}}{4\pi(s-1)} - \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{(e^{2\pi\nu}+1)(\nu^2+m^2+1/4)^s}\,, \tag{B.4}$$

where we have used that $\tanh(\pi\nu) = 1-2/[\exp(2\pi\nu)+1]$.

The divergence in the determinant is proportional to $-\zeta_B(0;m^2)$

$$-\zeta_B(0;m^2) = \frac{1}{4\pi}\left(m^2+\frac{1}{4}\right) + \frac{1}{48\pi} = -\frac{1}{4\pi}\left(\frac{1}{6}R^{(2)} - m^2\right)\,, \tag{B.5}$$

while the finite part

$$[\ln\det(-\Delta+m^2)]_{\rm fin} = -\zeta'_B(0;m^2) \tag{B.6}$$

is found, using (B.4), to be

$$\begin{aligned}\zeta'_B(0;m^2) = {} & \frac{1}{4\pi}\left(m^2+\frac{1}{4}\right)\left[\ln\left(m^2+\frac{1}{4}\right)-1\right] + \\ & + \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{e^{2\pi\nu}+1}\ln\left(\nu^2+m^2+\frac{1}{4}\right)\,.\end{aligned} \tag{B.7}$$

The coincidence limit of the propagator is (dropping divergences)[35]

$$G_B(0) = \zeta_B(-1;m^2) = -\frac{1}{4\pi}\ln\left(m^2+\frac{1}{4}\right) - \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{(e^{2\pi\nu}+1)(\nu^2+m^2+1/4)}$$
$$= -\frac{1}{2\pi}\psi\left(\frac{1}{2}+\sqrt{m^2+\frac{1}{4}}\right), \tag{B.8}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$. This is equal to $-\frac{d}{dm^2}\zeta'_B\left(0;m^2\right)$.

Since for $m^2 = -1/4$ we can evaluate $\zeta'(0;-1/4)$ explicitly,[36] we can write

$$\zeta'_B(0;m^2) = \frac{1}{24\pi}(1-\gamma-\ln\pi) + \frac{1}{4\pi^3}\zeta'_R(2) + \frac{1}{2\pi}\int_{-1/4}^{m^2} dx\,\psi\left(\frac{1}{2}+\sqrt{x+\frac{1}{4}}\right). \tag{B.9}$$

The density of states of the laplacian for a Majorana fermion is

$$\mu_F(\nu) = \pi\nu\coth(\pi\nu), \tag{B.10}$$

so that the ζ-function is

$$\zeta_F(s;m^2) = \frac{m^{2(1-s)}}{4\pi(s-1)} + \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{\left(e^{2\pi\nu}-1\right)\left(\nu^2+m^2\right)^s} \tag{B.11}$$

where we used that $\coth(\pi\nu) = 1 + 2/[\exp(2\pi\nu)-1]$. The divergence in the determinant is proportional to $-\zeta_F(0;m^2)$ and

$$-\zeta_F(0;m^2) = \frac{1}{4\pi}m^2 - \frac{1}{24\pi} = -\frac{1}{4\pi}\left(\frac{1}{6}R^{(2)} - \frac{1}{4}R^{(2)} - m^2\right). \tag{B.12}$$

This is the standard result for a 2-d fermion. As discussed in detail in the text (and in appendix C), in the case of GS fermion in the conformal gauge we should effectively multiply the $R^{(2)}$ term in $\zeta_F(0;m^2)$ by 4, ensuring the eventual cancellation of conformal anomalies and topological infinities.

The finite part of the determinant is

$$\ln\det\left(-D_F^2\right)_{\text{fin.}} = -\zeta'_F(0;m^2)$$
$$= -\frac{1}{4\pi}m^2\left(\ln m^2 - 1\right) + \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{e^{2\pi\nu}-1}\ln\left(\nu^2+m^2\right). \tag{B.13}$$

The derivative of ζ-function with respect to m^2

$$-\frac{d}{dm^2}\zeta'_F\left(0;m^2\right) = -\frac{1}{4\pi}\ln m^2 + \frac{1}{\pi}\int_0^\infty \frac{\nu d\nu}{\left(e^{2\pi\nu}-1\right)\left(\nu^2+m^2\right)}$$
$$= -\frac{1}{4\pi}\left(\frac{1}{|m|} + 2\psi(|m|)\right) \tag{B.14}$$

[35] To do the integral, we note that $\frac{1}{e^{2\pi\nu}+1} = \frac{1}{e^{2\pi\nu}-1} - \frac{2}{e^{4\pi\nu}-1}$, and use $\int_0^\infty \frac{\nu d\nu}{(e^{2\pi a\nu}-1)(\nu^2+x^2)} = \frac{1}{2}\left[\ln(ax) - \frac{1}{2ax} - \psi(ax)\right]$ and $2\psi(2x) - \psi(x) = \psi\left(x+1/2\right) + 2\ln 2$.

[36] To show this use $\int_0^\infty \frac{x\,dx}{e^x+1} = \frac{1}{2}\zeta_R(2)$, where ζ_R is the Riemann zeta function. and $\int_0^\infty \frac{x\,dx}{e^x+1}\ln x = \frac{1}{2}[(\psi(2)+\ln 2)\zeta_R(2) + \zeta'_R(2)]$, and the identities $\psi(2) = 1-\gamma$ and $\zeta_R(2) = \pi^2/$.

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is different from the fermion propagator from [37] (divided by $2m$). The expression (B.14) is obviously independent of the sign of the fermion mass term. However, supersymmetry relates fermions with opposite masses to scalars with different masses, and the supersymmetric regularization used in [37] gave a propagator that does depend on the sign. for the computation of the partition function, The prescription of [37] represents a different regularization scheme and thus leads to a different expression for the effective potential or partition function (see below).

Again, since we can calculate the finite part of the partition function explicitly at $m = 0$, we get

$$\zeta_F'(0;m^2) = -\frac{1}{12\pi}(1-\gamma-\ln 2\pi) - \frac{1}{2\pi^3}\zeta_R'(2) + \frac{|m|}{2\pi} + \frac{1}{2\pi}\int_0^{m^2} dx\,\psi(\sqrt{x})\,. \quad \text{(B.15)}$$

An $\mathcal{N} = 1$ supermultiplet in AdS_2 contains a fermion of mass $m_F = \mu$ and a boson of mass squared $m_B^2 = \mu^2 - \mu$ (4.15). One can combine (B.12) and (B.15) to get the partition function of a single multiplet. This can be written in terms of complicated special functions; for $\mu = \pm 1$ the numerical results are $\zeta_B'(0;0) - \zeta_F'(0;1) \sim 0.02688$ and $\zeta_B'(0;2) - \zeta_F'(0;1) \sim 0.05269$.

B.2 "Effective potential" in AdS_2

An alternative way to compute the partition function is to start with the Green's functions (defined in a way consistent with supersymmetry) and to integrate them over the mass parameter to obtain the effective potential as in [36, 35]. Following [37] for a multiplet of one boson and one fermion with masses related as in (4.15) we get

$$V_{\text{eff}}(\mu) = \frac{1}{2}\int_0^{m_B^2=\mu^2-\mu} dm^2\, G(m^2) - \frac{1}{2}\int_0^{m_F=\mu} dm\, 2m\, G(m^2-m)\,, \qquad \text{(B.16)}$$

where we normalized the effective potential to be zero in the case of the massless multiplet ($\mu = 0$). Here

$$\begin{aligned} G(m^2) &\equiv G_B(x,x|m^2) = \langle x|\,(-\nabla^2+m^2)^{-1}\,|x\rangle \\ &= \frac{1}{4\pi}\left[-\ln(c\Lambda^2) + 2\psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}+m^2}\right)\right], \end{aligned} \qquad \text{(B.17)}$$

where Λ is UV cutoff. In dimensional regularization ($d \to 2$) we get $\ln(c\Lambda^2) = \frac{2}{2-d} + \ln(4\pi\Lambda^2 a^2) - \gamma$. It is assumed that fermions are treated using dimensional reduction regularization which preserves supersymmetry, so that the fermionic Green function satisfies [37] $\operatorname{tr} G_F(x,x|\mu) = 2\mu G_B(x,x|\mu^2-\mu)$. Then

$$-\ln Z = W_0 + \int d^2\sigma\, V_{\text{eff}}(\mu)\,, \qquad \text{(B.18)}$$

where W_0 is the contribution of the massless multiplet that can be found by integrating the conformal anomaly. This gives an alternative way to compute the partition function.

In our case of 3 multiplets with $\mu = -1$ and 5 multiplets with $\mu = 1$ we get a logarithmically divergent result: the constant (m-independent) part of $G(m^2)$ is multiplied by

$$3\left[-\frac{1}{2}\int_2^0 dm^2 + \frac{1}{2}\int_{-1}^0 dm\,2m\right] + 5\left[-\frac{1}{2}\int_0^0 dm^2 + \frac{1}{2}\int_1^0 dm\,2m\right] = \frac{1}{2}(3-5) = -1\,. \tag{B.19}$$

The finite part is also non-zero. For a single multiplet

$$[V_{\rm eff}(\mu)]_{\rm fin} = -\frac{1}{4\pi}\Big[\int_{\mu^2-\mu}^0 dm^2\,\psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}+m^2}\right) - \int_\mu^0 dm\,2m\psi\left(\frac{1}{2}+\left|m-\frac{1}{2}\right|\right)\Big] = \frac{1}{4\pi}\int_\mu^0 dm\psi\left(\frac{1}{2}+\left|m-\frac{1}{2}\right|\right), \tag{B.20}$$

where we have changed the integration variable in the first term ($m^2 \to m^2 - m$).[37] For $\mu = -1$ we get explicitly

$$[V_{\rm eff}(-1)]_{\rm fin} = \frac{1}{4\pi}\int_{-1}^0 dm\,\psi(1-m) = \frac{1}{4\pi}\int_1^2 dm\,\psi(m) = 0\,. \tag{B.21}$$

For $\mu = 1$ we get zero bosonic contribution and thus

$$\begin{aligned}[V_{\rm eff}(1)]_{\rm fin} &= -\frac{1}{4\pi}\int_0^1 dm\,2m\psi\left(\frac{1}{2}+\left|m-\frac{1}{2}\right|\right)\\ &= -\frac{1}{4\pi}\int_{1/2}^1 dm\,2\psi(m) = \frac{1}{4\pi}\ln\pi\,,\end{aligned} \tag{B.22}$$

where we split the integral into two parts and changed the variable.

For $3+5$ multiplets we get a non-zero result (note that the first term is actually zero)

$$3[V_{\rm eff}(-1)]_{\rm fin} + 5[V_{\rm eff}(1)]_{\rm fin} = \frac{5}{4\pi}\ln\pi\,. \tag{B.23}$$

The contribution of the 8 massless multiplets can be found in the conformal frame $ds^2 = w^{-2}(dt^2+dw^2)$ to be

$$\begin{aligned}W_0 &= -8\times\left(1+\frac{1}{2}\right)\times\frac{1}{24\pi}\int d^2\sigma\sqrt{g}g^{ij}\partial_i\rho\partial_j\rho\\ &= -\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\,,\end{aligned} \tag{B.24}$$

where $g^{ww}\partial_w\rho\partial_w\rho = 1$, $\rho = -\ln w$.

[37] The observation that for a supermultiplet the two terms combine in this way was made in [37] and implicitly in [30]. Notice that since $\int_b^a dm\,\psi(m) = \ln\Gamma(a) - \ln\Gamma(b)$, the resulting integral is easily computable by splitting the interval and changing the variables.

C. Comments on 2-d determinants

In the main text we have performed (following earlier discussions of the GS string in [19, 13, 16, 14, 15, 17]) a local rotation to put the GS action (3.4) in a "2-d fermion in curved 2-d space" form. In general, the resulting jacobian is non-trivial and is given by a Polyakov-Wiegmann [29] expression which is a U(1) WZ action.

In the case when 2-d metric is kept independent the account of the contribution of the rotation jacobian is crucial in order to show that the conformal anomaly of a GS fermion is 4 times the naive anomaly of a 2-d fermion — as needed to cancel the conformal anomaly of GS string in flat space. As was already mentioned in the text, the cancellation of (dilatonic part of) the conformal anomaly in the one-loop approximation we considered is exactly the same as in flat space (the curved space-time background changes the $O(R^{(2)})$ conformal anomaly only starting with the 2-loop approximation [53]).

Let us summarize some basic facts about these determinants. They are always defined modulo local counterterms of background fields which are to be chosen consistent with the symmetries that are to be preserved (see [14]). For generality, we consider the chiral case, the non-chiral one is just a combination of the two chiral ones. The standard 2-d Weyl spinor operator is (we use the euclidean notation where $\tau_{1,2} = \sigma_{1,2}$ are the Pauli matrices; here $k, n, m = 1, 2$)

$$D(e) = \frac{1}{2} i(1 - \tau_3) e^k_\alpha \tau^\alpha (\partial_k + i\tau_3 w_k) \, , \qquad \omega^{\alpha\beta}_k = 2\epsilon^{\alpha\beta} w_k \, . \tag{C.1}$$

Then

$$\ln \det D(e) = -\frac{i}{12\pi} I(iw) = \frac{1}{2 \times 96\pi} \int R^{(2)} \left(-\nabla^2\right)^{-1} \left(R^{(2)} - 4i\nabla_k w^k\right) . \tag{C.2}$$

In the case of an abelian gauge field background in flat space

$$I(A) = W(T_A, A) = \int dT_A T_A^{-1} \wedge A \, , \qquad (\sqrt{g} g^{mn} + i\epsilon^{mn})(\partial_n + A_n) T_A = 0 \, . \tag{C.3}$$

In conformal coordinates $A_z = -\partial_z \lambda$, $\lambda = \ln T_A$, so that $I(A) = \int \partial_z \lambda (\bar\partial_z \lambda - A_{\bar z})$, which is different from usual abelian expression $\int \partial\lambda\bar\partial\lambda$ by the local counterterm $\int A_z A_{\bar z}$.[38] The gauge field dependent WZ term in the Dirac operator case is propor-

[38] An equivalent (up to a local counterterm) expression is

$$\ln \det D = -\frac{1}{3} Z(w) + Z(A) \, , \qquad Z(B) = -\frac{1}{4\pi} \int \epsilon^{mn} \partial_m B_n \left(-\nabla^2\right)^{-1} \left[\epsilon^{mn} \partial_m B_n - i\nabla_m B^m\right] .$$

Using the conformal gauge and definitions $g_{mn} = e^{2\rho}\delta_{mn}$, $w_m = \partial_m \lambda + \frac{1}{2}\epsilon_{mn}\partial^n \rho$, $A_m = \partial_m a + \epsilon_{mn}\partial^n b$, we get

$$\ln \det D(e, A) = \frac{1}{4\pi} \int d^2\sigma \left[\frac{1}{12} \partial^m \rho \left(\partial_m \rho + \frac{1}{2} i\partial_m \lambda\right) - \partial^m b \left(\partial_m b + \frac{1}{2} i\partial_m a\right)\right] .$$

tional to $I(A)+I(\bar{A})$, or

$$i\int\left(A\frac{\bar{\partial}}{\partial}A+\bar{A}\frac{\partial}{\bar{\partial}}\bar{A}\right), \tag{C.4}$$

or, after adding a local $A\bar{A}$ term, $-i\int(\partial\bar{A}-\bar{\partial}A)\frac{1}{\partial\bar{\partial}}(\partial\bar{A}-\bar{\partial}A)$.

For a Majorana spinor on a curved background only the first (Polyakov) conformal anomaly term is present in (C.2) (with coefficient which is 1/2 of the scalar one), while the imaginary Lorentz-anomaly term cancels out. The gravitational WZW action I is simply $\sim\int\partial^k b\partial_k b$, where $w^k=i\epsilon^{kn}\partial_n b$, up to a local counterterm $\int w^k w_k$.

If there is also some internal connection acting on flavor indices of fermions, then under the chiral projector in 2 dimensions it can be formally rotated away (this is clear in conformal coordinates)

$$D(e,A)=\frac{1}{2}i(1-\tau_3)e^m_\alpha\tau^\alpha(\partial_m+i\tau_3 w_m+A_m)=T_A D(e)T_A^{-1}, \tag{C.5}$$

where T_A is a local rotation "eliminating" A_m, and

$$\ln\det D(e,A)=-\frac{iN}{12\pi}I(iw)+\frac{i}{4\pi}I(A). \tag{C.6}$$

Now, consider a more general case we are actually interested in

$$D(\rho,B)=\frac{1}{2}i(1-\rho_3)\rho^m(\partial_m+B_m), \tag{C.7}$$

where $\rho_m=\rho_m(x)$ is an arbitrary $2N\times 2N$ representation of the 2-d Dirac algebra for some metric $g_{mn}(x)$, satisfying

$$\rho_m\rho_n=g_{mn}+ie_{mn}\rho_3, \qquad e^{mn}=\frac{1}{\sqrt{g}}\epsilon^{mn}. \tag{C.8}$$

The condition on the connection B_m is

$$\partial_m\rho_n+[B_m,\rho_n]-\Gamma^k_{mn}\rho_k=0. \tag{C.9}$$

Since $g_{mn}=e^\alpha_m e^\beta_n\delta_{\alpha\beta}$, there exists a local rotation S that transforms ρ_m into the constant 2×2 Pauli matrices times $N\times N$ unit matrix times the zweibein e^α_n

$$S\rho_n S^{-1}=\tau_\alpha\times I\ e^\alpha_n. \tag{C.10}$$

Then

$$\begin{aligned} SD(\rho,B)S^{-1} &= D(e,A), \\ \partial_m+i\tau_3 w_m+A_m &= S(\partial_m+B_m)S^{-1}, \end{aligned} \tag{C.11}$$

and hence [14]

$$\ln\det D(\rho,B)=-\frac{iN}{3\pi}I(iw)+\frac{i}{8\pi}I(B). \tag{C.12}$$

Note that for $B_m = iw_m$ (with extra 2 flavors, i.e. $I(B) = 2I(iw)$) and $N = 1$ we get back to (C.2), i.e. $-\frac{i}{12\pi}I(iw)$.

For the flat target space [13] (B_m comes from integration by parts as required for self-adjointness of the Dirac operator)

$$\rho_n = \partial_n x^a \Gamma_a \,, \qquad B_m = \frac{1}{2}\rho_3 \partial_m \rho_3 = -\frac{1}{2}\rho_n \nabla_m \rho^n \,. \tag{C.13}$$

In the case of the non-chiral Dirac fermions (where there is no problem with a definition of the determinant of $D(\rho, B)$ discussed in [14]) we get simply

$$\ln \det D(\rho, B) = \frac{N}{24\pi} \int R^{(2)} \left(-\nabla^2\right)^{-1} R^{(2)} + \frac{i}{8\pi}[I(B) - \tilde{I}(B)] \,, \tag{C.14}$$

where $\tilde{I}$ is defined by (C.3) with parity-reflected condition. In the abelian case $I(B) - \tilde{I}(B) = 0$, up to a local counterterm. Notice that the conformal anomaly term (here we discuss the Dirac spinor, so that it is twice that of a Majorana GS fermion) is 4 times bigger than for the usual 2-d Dirac fermion.

D. Superstring partition function in $AdS_3 \times S^3$ with RR flux

The same calculation can be done in the case of a superstring in the $AdS_3 \times S^3 \times T^4$ with RR background. The corresponding GS action has the form very similar to $AdS_5 \times S^5$ one and was discussed in [54]. Since all the classical solutions discussed above depend on only three of the AdS_5 coordinates, they can be directly embedded in AdS_3 and are still minimal surfaces.

There are some differences in treating the quadratic fluctuations in $AdS_3 \times S^3 \times T^4$:

- there are no massive fluctuations in the x^2, x^3 directions which are replaced by extra two massless fluctuations in the toroidal T^4 directions.
- only half of the 8 effective 2-d fermions get σ_3 mass term in (4.11), (5.17) and (6.29). In the present case it is natural to split the Γ-matrices in (3+3)+4 = 6 + 4 way, $\Gamma_a = \gamma_a \times I_4$, $a = 0, 1, \ldots, 5$, where γ_a are 6-d 8×8 matrices. The 'mass term' in the covariant derivative (3.6) and in the string action now originates from the sum of the electric and magnetic RR 3-form field strengths, $\partial x^\mu \partial x^\nu \bar{\theta}\Gamma_\nu \Gamma^{abc}\Gamma_\mu (F_{abc} + F^*_{abc})$. The combination of the field strengths produces the $(1 + \Gamma_7)$ projection operator, so that only half of the 6-d fermions get a mass term.

For example, in the case of the straight line the minimal surface is still AdS_2, but the set of fluctuation fields is different. There are 7 massless bosons, one mass 2 boson, four massless fermions and four mass 1 fermions. They form one $\mathcal{N} = 4$

multiplet (the dimensional reduction of the $\mathcal{N} = 2$ vector multiplet in $D = 4$) with three massless and one massive boson and four massive fermions. There are 4 other massless bosons and four massless fermions which can also combine into $\mathcal{N} = 4$ multiplet. One can also check that the vacuum energy as defined by the zeta function again vanishes for this combination of fields. The same conformal anomaly calculation as in section 3.4 gives that $Z = 1$.

In the case of the general bent string configuration (parallel Wilson lines), the analog of the $AdS_5 \times S^5$ partition function (6.54) takes the form (in static gauge)

$$Z^{||}_{AdS_3\times S^3\times T^4} = \frac{\det^{4/2}\left(-\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1\right)\det^{4/2}\left(-\hat{\nabla}^2 + \frac{1}{4}R^{(2)}\right)}{\det^{1/2}\left(-\nabla^2 + R^{(2)} + 4\right)\det^{7/2}\left(-\nabla^2\right)}. \tag{D.1}$$

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